Topological Complexity of Configuration Spaces

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To my parents

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Abstract

In this thesis we study the homotopy invariant $\mathsf{TC}(X)$; the topological complexity of a space X. This invariant, introduced by Farber in [15], was originally motivated by a problem in Robotics; the motion planning problem. We study relations between the topological complexity of a space and its fundamental group, namely when the fundamental group is "small", *i.e.* either has small order or small cohomological dimension. We also apply the navigation functions technique introduced in [20] to the study of the topological complexity of projective and lens spaces. In particular, we introduce a class of navigation functions on projective and lens spaces. It is known ([25]) that the topological complexity of a real projective space equals one plus its immersion dimension. A similar approach to the immersion dimension of some lens spaces has been suggested in [31]. Finally, we study the topological complexity (and other invariants) of random right-angled Artin groups, *i.e.* the stochastic behaviour of the topological complexity of Eilenberg-MacLane spaces of type $K(G, 1)$, where G is a right-angled Artin group associated to a random graph.

Declaration

The work in this thesis is based on research carried out in the Pure Mathematics Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

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Contents

Introduction

The *motion planning problem* is a central theme in Robotics. Consider a mobile robot in a room with obstacles. The robot must move from one side of the room to the other avoiding the obstacles. The basic solution to this problem is given by a choice of a path connecting the robot to its final goal. One may also want to control simultaneously several robots avoiding collisions with the obstacles and with themselves. A main reference for a deep exposition on the motion planning problem is [38].

In a more general setting one would like to solve the motion planning problem for a configuration space X associated with a given mechanical system S. Solving this problem means creating an algorithm which produces a motion connecting any two given states of the configuration space. In this thesis the configuration space is always assumed to be path-connected.

The concept of configuration space is common to both Topology and Robotics. Configuration spaces of real physical/mechanical systems have often interesting topology. Besides, one is often able to predict instabilities in the system by studying the topology of its configuration space [1].

In [15], Farber introduced the concept of Topological Complexity of a configuration space X, denoted by $\mathsf{TC}(X)$. The number $\mathsf{TC}(X)$ is a numerical homotopy invariant of configuration spaces and in a specific sense measures the instability/discontinuity of the motion planning problem.

The main goal of this thesis is to study properties of the invariant $\mathsf{TC}(X)$. Since it is an homotopy invariant it is an interesting object to study in Topology. In fact, the notion of topological complexity has been proven to be linked with other concepts in Topology. As an example we may mention a result of Farber, Tabachnikov and Yuzvinsky:

Theorem ([25])**.** *For any $n \geq 1$ except $n = 1, 3, 7$, the number $\mathsf{TC}(\mathbb{R}P^n)$ equals the smallest dimension k for which $\mathbb{R}P^n$ immerses into $\mathbb{R}^{k-1}$.*

This thesis can be separated into two parts. The first part contains Chapters 1, and 2; it serves as an introduction to the main concept of the thesis, the notion of Topological Complexity, and gives an overview on state of the art. With the exception of Corollary 2.3.3, no new results are introduced in this part of the thesis. The second part, composed of the remaining three chapters, contributes new results to this area of research. Chapters 3 and 5 survey the results of the articles [7] and [8], respectively. The results in Chapter 4 are also original.

We will now offer to the reader a more detailed picture on the structure of the thesis. Chapter 1 introduces the concept of configuration space and describes some of the configuration spaces relevant to robotics. The basic motion planning problem, known as the *Piano movers' problem*, is introduced. A main reference is [38].

In Chapter 2 we survey most of the relevant results known in the subject of topological complexity. The basic techniques to compute the number $\mathsf{TC}(X)$ are presented in Section 2.4. Section 2.5 describes the topological complexity of several configuration spaces. For a more developed exposition we refer to [20]

Chapter 3 is an exposition of the joint work with M. Farber, supervisor to the author of this thesis, developed in [7]. There we established upper bounds for $\mathsf{TC}(X)$, when the fundamental group $\pi_1(X)$ is "small", *i.e.* it is either cyclic of small order or has small cohomological dimension.

In Chapter 4 we study the concept of *navigation function* on a manifold M; navigation functions were introduced in [20]. These are non-negative Morse-Bott functions on $M \times M$ which are valued zero exactly on the diagonal $\Delta_M = \{(x, y) \in M \times M|\ x = y\}$. The connection with topological complexity is given by Theorem 4.1.1. We introduce a class of navigation functions on lens (and projective) spaces and describe a thorough computation of the respective critical submanifolds.

Chapter 5 covers the joint work with M. Farber exposed in [8]. There we show that the topological complexity of a random right angled Artin group assumes at most three values, with high probability. Random spaces arise naturally as configuration spaces of large or partially unknown configuration spaces. The topological complexity of a right angled Artin group was first studied in [9].

Chapter 1

Configuration Spaces in Robotics

The **configuration space** of a given physical/mechanical system is the space of all possible states of that system. A **state** is a description of a specific system configuration.

In this chapter we describe some configuration spaces which appear in Robotics. These are configuration spaces associated with a given automated mechanical system. Studying the topology of such spaces may help to predict instabilities in the motion of the system. Our main focus will be to study the topology of configuration spaces arising from mechanical systems. Often the system will consist of one or more particles moving in a certain space and subject to a number of restrictions. A classic example is the *Piano movers' problem* [38].

1.1 Examples of configuration spaces

Example 1.1.1 (*Piano movers' problem*)**.** We wish to move a "piano" from one point of a room to another point without colliding with a certain number of objects; see Figure 1.1. One way to describe a specific state of the system would be to

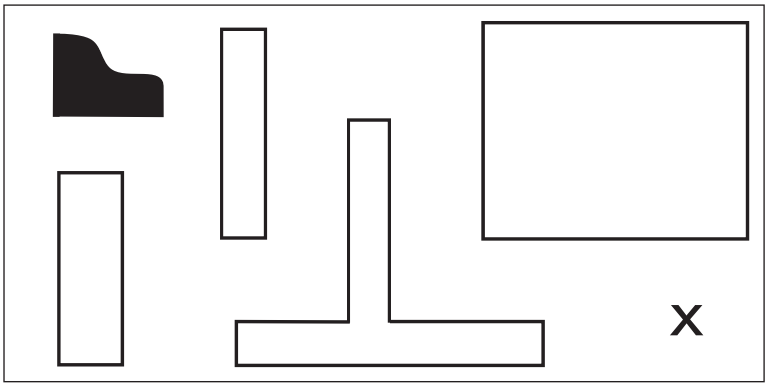

Figure 1.1: Piano Movers' problem.

determine the coordinates of the piano's center and its orientation. One can only have states for which the piano does not intersect with any of the obstacles. The associated configuration space is a 3-dimensional space with a possibly complicated geometry. One may even add more complexity to the system by adding moving obstacles. The *asteroid avoidance problem* is the problem of planning the motion of an object in a 2-dimensional or 3-dimensonal space while avoiding moving obstacles; see [38].

Example 1.1.2 (*Robot arm*)**.** A typical mechanical system is the *robot arm.* The arm consists of a certain number n of rigid bars in the plane attached by revolving joints as illustrated in the Figure 1.2.

One way to describe a specific position of the arm is to determine the angles formed by the bars at the revolving joints (the initial angle being formed by the first bar and the horizontal axis). Knowing the angles determines completely the position of the arm. Notice that we are allowing self-intersections of the arm. In this case the configuration space associated to the robot arm is the n-torus

$$X = S^1 \times \ldots \times S^1.$$

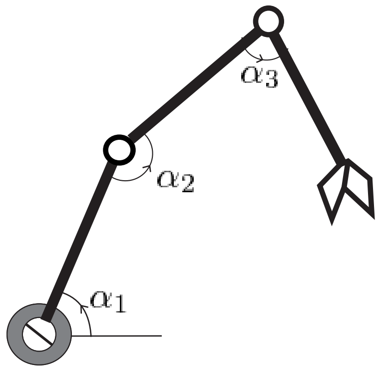

Figure 1.2: Robot arm.

A variation of the robot arm system can be obtained by requiring that the last bar is connected to the first by a revolving joint. This mechanism is known as a mechanical linkage.

Example 1.1.3 (*Linkages*)**.** A (planar) *mechanical linkage* is a closed sequence of rigid bars in the plane, with possibly different fixed lengths, connected by revolving joints; see figure 1.3. We assume that one joint is located at the origin and that two shapes are the same if there is a rotation of the plane transforming one into the other. The configuration space of this system is the space of closed polygonal shapes with possible self-intersections.

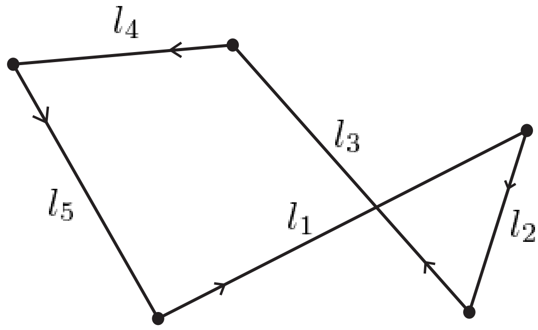

Figure 1.3: Linkage with 5 bars.

The *length vector* of the linkage is a vector that encodes the lengths of the bars in the mechanism. Given a length vector

$$l = (l_1, \ldots, l_n) \in \mathbb{R}^n_+,\ l_1, \ldots, l_n > 0,$$

the configuration space of all the possible linkages in the plane is the moduli space

$$M_l = \{(u_1, \ldots, u_n) \in S^1 \times \ldots \times S^1 |\ \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{SO}(2).$$

Here the group of rotations $SO(2)$ acts diagonally on the vector $(u_1, \ldots, u_n)$, *i.e* acts identically in each entry of the length vector. The length vector $l = (l_1, \ldots, l_n)$ is called *generic* if

$$\sum_{i=1}^{n} a_i l_i \neq 0,\ \text{for } a_i = \pm 1.$$

It is well known that if l is generic then the space M_l is an orientable manifold of dimension $n-3$; a detailed survey about these spaces can be found in Chapter 1 of [20].

It may happen that the parameters of a mechanical system are partially unknown. One may also consider systems with a large number of parameters. In this case the exact geometry of the configuration system is unknown. However, in some cases one can predict with high confidence some aspects of the topology of the configuration space. In [24] the authors studied the Betti numbers of random linkages, *i.e.* linkages with the lengths of the bars viewed as random variables.

Example 1.1.4 (*Projective spaces*)**.** Consider a rigid bar revolving around its middle point in the Euclidean space $\mathbb{R}^{n+1}$. The configuration space of this system is the projective space $\mathbb{R}P^n$.

We now introduce a system with multiple objects.

Example 1.1.5 (*Particles avoiding collisions*)**.** Let X be a finite simplicial polyhedron with n distinct particles moving in X. The space

$$F(X,n) = \{(x_1,\dots,x_n)|\ x_i \neq x_j \text{ for } i \neq j\}$$

is the configuration space of n particles moving in X avoiding collisions. One may also consider the configuration space

$$B(X,n) = F(X,n)/\Sigma_n,$$

where Σ_n is the symmetric group of degree n and acts freely on $F(X,n)$ by permutation of the particles. This is the configuration space of n unordered particles moving in the space X avoiding collisions.

Usually one considers two special cases: $X = \Gamma$ where Γ is a connected graph or X is the Euclidean space $\mathbb{R}^k$.

In [28] R. Ghrist introduces the space $F(\Gamma,n)$ as the configuration space for a system with n robots working in a network of a factory floor. This application serves as an example on how studying the topology of the space $F(\Gamma,n)$ may help to predict the complexity of the multiple robot control problem.

The spaces $F(\mathbb{R}^k,n)$, introduced by Fadell and Newirth in [12], have been widely studied in Topology. For a general exposition consult [13]. These spaces have strong connections with the theory of braid groups. For example, the braid group with n strings, B_n, is the fundamental group of the space $B(\mathbb{R}^2,n)$.

The space $F(X,n)$ where X is an algebraic variety or an orientable manifold has been studies by Totaro in [47]. There Totaro studied the cohomological algebra of these spaces.

Curiously, the homotopy type of $F(X,n)$ is not an homotopy invariant even for manifolds. In [39], Salvatore and Longoni show that there are two homotopy

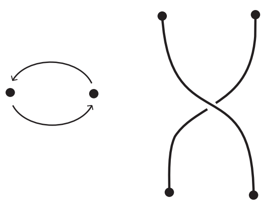

Figure 1.4: On the left, two particles interchanging position without collision. On the right, the respective element of the braid group B_2.

equivalent spaces X_1 and X_2 such that $F(X_1, n)$ is not homotopy equivalent to $F(X_2, n)$. The spaces used in that proof were two homotopy equivalent but not homeomorphic lens spaces with fundamental group $\mathbb{Z}_7$.

Chapter 2

Topological Complexity of Configuration Spaces

In this chapter we introduce the concept of Topological Complexity, which arises from the motion planning problem discussed in the previous chapter. The topological complexity of a space X, denoted by $\mathsf{TC}(X)$, is an homotopy invariant introduced by M. Farber in a series of papers; see [15], [16] and [25]. This chapter intends to serve as an introduction to the notion of topological complexity. The book [20] is recommended as a survey covering most of the known results.

2.1 Motion planning from a topological viewpoint

Consider a mechanical system S with associated configuration space X. We will always assume that X is path-connected. Each point of X is a state of the system S. A continuous path in X corresponds to a continuous motion between two states of the system. A **motion planning algorithm** is defined by assigning to each input (S_1, S_2), where S_1 and S_2 are states of S (*i.e.*, points in X), an output γ, where γ is a continuous path in X that connects S_1 to S_2.

Let PX be the free path space of X, the set of all continuous paths $\gamma : [0,1] \to X$, equipped with the compact-open topology. The **path space fibration** of X is the fibration

$$\mathfrak{p} : PX \to X \times X \tag{2.1}$$

given by $\mathfrak{p}(\gamma) = (\gamma(0), \gamma(1))$. One can describe a motion planning algorithm as a map $s : X \times X \to PX$ such that $\mathfrak{p} \circ s = \mathrm{Id}_{X\times X}$. Hence, a motion planning algorithm (or simply motion planner) is a section of $\mathfrak{p}$.

Most spaces do not admit any *continuous* motion planning algorithm. In fact, assume there is a continuous section $s : X \times X \to PX$ of the path space fibration $\mathfrak{p} : PX \to X \times X$. Fix $B \in X$ and set $S(x,t) = s(x,B)(t)$. Then $S(x,0) = x$ and $S(x,1) = B$. Moreover, since s is a continuous map, S is a deformation retract of X to a point. Hence, only contractible spaces may admit continuous motion planning algorithms. The converse is also true; see [15] Theorem 1.

Lemma 2.1.1 ([15])**.** *A continuous motion planner in the space X exists if and only if X is contractible.*

2.2 Topological complexity of a space

We have just seen that a motion planner on a space X will usually have some amount of discontinuity. An instrument to measure this discontinuity is the topological complexity of the space X. A main reason of interest in this quantity is that it is a homotopy invariant of the configuration space.

In this thesis will always assume that X is a path-connected topological space.

Definition 2.2.1. *The* ***topological complexity*** *of the space X, $\mathsf{TC}(X)$, is the minimal number k such that there exists an open cover $X \times X = U_1 \cup \ldots \cup U_k$ with the property that each U_i admits a continuous motion planner $s_i : U_i \to PX$.*

Remark 2.2.1. In view of Lemma 2.1.1 one could ask if the sets U_i are always contractible or null-homotopic. This is not true since for a polyhedron X a continuous motion planner always exists over some neighborhood of the *diagonal of $X \times X$,*

$$\Delta_X = \{(x, x) |\ x \in X\} \subset X \times X,$$

which in general is not contractible in $X \times X$.

Definition 2.2.1 is a particular instance of the notion of genus of a fibration, introduced by A. Schwarz in the seminal paper [45].

Definition 2.2.2. *The* ***genus*** *of a Serre fibration (also known as* ***Schwarz genus****) $p : E \to B$ is the minimal integer k such that there exists an open cover of B with k elements each of which admiting a continuous section of the corresponding restriction of p.*

One can define $\mathsf{TC}(X)$ in terms of the Schwarz genus of the fibration $\mathfrak{p}$ defined in (2.1).

Definition 2.2.3. *The topological complexity of X, $\mathsf{TC}(X)$, is the genus of the fibration $\mathfrak{p}$.*

One of the main properties of the Schwarz genus is the homotopy invariance, *i.e.*, if $p : E \to B$ is a fibration and $h : B' \to B$ a homotopy equivalence, then $genus(p) = genus(p')$, where p' is the induced fibration of h by p.

Proposition 2.2.1. *The topological complexity of a space is a homotopy invariant.*

Denote by P_0X the restriction of PX to paths starting at a fixed point x. Consider the fibration $\mathfrak{p}_0 : P_0X \to X$, where $\mathfrak{p}_0(\gamma) = \gamma(0)$. The Schwarz genus of $\mathfrak{p}_0$ is the classic **Lusternik-Schnirelmann category** of the space X, which we denote as $\mathsf{cat}(X)$[1]. Consequently, also the Lusternik-Schnirelmann category is a numerical homotopy invariant of spaces. For an introduction to Lusternik-Schnirelmann category consult [10].

A continuous section of $\mathfrak{p}_0$ over a set $U \subset X$ is a continuous map $r : U \to P_0X$, that to a point of $u \in U$ assignes a path $r(u)$ which starts at the fixed point x and ends at u. Let $R : U \times [0,1] \to X$ be the map given by $R(u,t) = r(u)(t)$. Clearly R is continuous and a deformation retract of U onto a point, hence U is null-homotopic in X.

The invariants $\mathsf{TC}(X)$ and $\mathsf{cat}(X)$ are naturally correlated as they are the genus of related fibrations. The fibration $\mathfrak{p}_0 : P_0X \to X$ is the pullback of the path fibration $\mathfrak{p} : PX \to X \times X$ by the inclusion $* \times X \to X \times X$.

Proposition 2.2.2. *For a path-connected topological space X it holds*

$$\mathsf{cat}(X) \leq \mathsf{TC}(X) \leq \mathsf{cat}(X \times X) \leq 2\mathsf{cat}(X) - 1. \tag{2.2}$$

A proof can be found in [15], Theorem 5.

In [19] Farber introduces several descriptions for the topological complexity of a space and proves they are equivalent for a vast class of spaces; *e.g.* the class of finite simplicial polyhedra (subspaces of a Euclidean space which are homeomorphic to the underlying space of a finite simplicial complex). We present below one of those alternative descriptions.

[1] We warn the reader that some authors define $\mathsf{cat}(\mathrm{X})$ as $genus(\mathfrak{p}_0) - 1$.

Definition 2.2.4. *A motion planning algorithm $s : X \times X \to PX$ is called* ***tame*** *if $X \times X$ can be split into finitely many sets $X \times X = F_1 \cup F_2 \cup \ldots \cup F_k$ such that*

1. *$s|_{F_i} : F_i \to PX$ is continuous, $i = 1 \ldots k$,*

2. *$F_i \cap F_j = \emptyset$, whenever $i \neq j$,*

3. *Each F_i is an Euclidean Neighboorhood Retract (ENR), i.e. if it can be embedded into an Euclidean space $X \subset \mathbb{R}^k$ such that, for some open neighborhood $X \subset U \subset \mathbb{R}^k$, there exists a retraction $r : U \to X$, $r|_X = 1_X$.*

A cover with the above properties is called a ***tame cover***[2].

Definition 2.2.5. *The topological complexity of a path-connected topological space X is the minimal k such that X possesses a tame cover with k elements.*

Example 2.2.3. The simplest non-contractible spaces are spheres. Using Definition 2.2.5, a simple argument shows that $\mathsf{TC}(S^n) = 2$ for n odd and $\mathsf{TC}(S^n) \leq 3$ for n even.

- For n odd let $U_1 \subset S^n \times S^n$ be the set of all pairs (A, B) with $A \neq -B$. Set $s_1 : U_1 \to PS^n$ to assign to a pair (A, B) the path connecting A to B through the unique shortest geodesic arc (assume parameterized by arc-length). This defines a continuous motion planner over U_1. Since n is odd one can has a non-vanishing tangent vector field X over S^n. Take U_2 to be the set of all pairs $(A, -A)$, where $A \in S^n$. To such a pair assign the path that runs through the unique semi-circle connecting A to $-A$ which has the direction of $X(A)$ at A and is parameterized by arc-length. Combining with Lemma 2.1.1, it follows that $\mathsf{TC}(S^n) = 2$ for n odd.

[2] We warn the reader of a homonymous notion in the theory of schemes.

- In the case that n is even, we may proceed analogously for the domain $U_1 = \{(A, B) \subset S^n \times S^n : A \neq -B\}$. On the other hand, any tangent vector field on S^n must have at least one zero. However, one can always find a tangent vector field that possesses a single zero A_0. Set $U_2 = \{(A, -A) : A \neq A_0\}$ and define a section over U_2 identical to the one defined for n odd. To the remaining point $(A_0, -A_0)$, we may assign an arbitrary path connecting A_0 to A_0.

Whereas Definition 2.2.1 allows us to estimate $\mathsf{TC}(X)$ through the use of algebraic topology tools, as we will see later, Definition 2.2.5 arises naturally from *real world* motion planning algorithms and has a greater geometrical flavor. Except when explicit otherwise, we will adopt Definition 2.2.1 as the definition of $\mathsf{TC}(X)$. In [19] Farber also presents a characterization of TC through random motion planning algorithms.

2.3 Relative topological complexity

The notion of relative topological complexity was introduced by Farber in [20]. As we have seen, $\mathsf{TC}(X)$ is defined as the Schwarz genus of the path fibration $\mathfrak{p} : PX \to X \times X$. If we consider a subset $A \subseteq X \times X$, the relative topological complexity of A is the genus of the restriction of $\mathfrak{p}$ to A.

Definition 2.3.1. *Let X be a topological space and $A \subseteq X \times X$. Let $P_A X \subset PX$ be the space of all paths in X with endpoints in A and let $\mathfrak{p}_A : P_A X \to A$ be the restriction of $\mathfrak{p}$ to A. Then $\mathsf{TC}_X(A)$ is the smallest integer k such that there is an open cover $\{U_i\}_{1 \leq i \leq k}$ of A with the property that for each i there is a continuous section of $\mathfrak{p}_{U_i}$.*

Remark 2.3.1. Suppose that A and B are E.N.R. sets such that $A \cup B = X \times X$. Then

$$\mathsf{TC}(X) \leq \mathsf{TC}_X(A) + \mathsf{TC}_X(B).$$

Notice also that for any subset $A \subset X \times X$ one has $\mathsf{TC}_X(A) \leq \mathsf{TC}(A)$. This is due to the natural inclusion $PA \subset P_A X$.

Remark 2.3.2. Clearly the number $\mathsf{TC}(X)$ is the minimal integer k for which there is an open cover $U_1, \ldots, U_k$ of $X \times X$ with the property that $\mathsf{TC}_X(U_i) = 1$ for $1 \leq i \leq k$.

A property of relative topological complexity is the following:

Lemma 2.3.1 ([20])**.** *Suppose that the sets $A \subset B \subset X \times X$ are such that B can be deformed into A inside of $X \times X$. Then*

$$\mathsf{TC}_X(A) = \mathsf{TC}_X(B).$$

The next lemma describes which subsets have minimal relative topological complexity.

Lemma 2.3.2. *Let $A \subseteq X \times X$. The following statements are equivalent*

- $\mathsf{TC}_X(A) = 1$*;*
- *the two projections of A on each of the factors of $X \times X$ are homotopic;*
- *the inclusion $A \to X \times X$ is homotopic to a map $A \to \Delta_X$, where Δ_X denotes the diagonal of $X \times X$.*

As an illustration we present the Corollary below.

Corollary 2.3.3. *Let M be the Moebius band and $B = \partial M$ the respective boundary. One has*

$$\mathsf{TC}_M(B \times B) = 2.$$

Proof. One has $M \sim S^1$ and $B \sim S^1$ (where $\sim$ denotes homotopy equivalence), where B is included in M by the map $S^1 \xrightarrow{i} S^1$ given by $i(z) = z^2$, where $z \in S^1$. By the previous lemma $\mathsf{TC}_M(B \times B) = 1$ if and only if the map $S^1 \times S^1 \xrightarrow{p_1} S^1$ is homotopic to the map $S^1 \times S^1 \xrightarrow{p_2} S^1$, where

$$p_1(z_1, z_2) = z_1^2 \quad \text{and} \quad p_2(z_1, z_2) = z_2^2.$$

This is clearly not true since the maps p_1 and p_2 can be identified with loops which are not homotopic in the torus $S^1 \times S^1$.

On the other hand, by Remark 2.3.1 one has

$$\mathsf{TC}_M(B \times B) \leq \mathsf{TC}(M) = \mathsf{TC}(S^1) = 2.$$

Hence $\mathsf{TC}_M(B \times B) = 2$. □

2.4 Bounds on $\mathsf{TC}(X)$

In this section we describe some of the methods for determining the topological complexity of a configuration space X. The cohomology of the space X will play a central role in the methods for obtaining lower bounds for $\mathsf{TC}(X)$.

2.4.1 Upper bounds

Schwarz genus properties imply the following result:

Theorem 2.4.1 ([16]). *If X is an r-connected simplicial polyhedron with covering dimension $\dim X$, then*

$$\mathsf{TC}(X) < \frac{2\dim X + 1}{r+1} + 1. \tag{2.3}$$

In particular we have the general bound

$$\mathsf{TC}(X) \leq 2\dim X + 1. \tag{2.4}$$

The topological complexity of the product of spaces is at most additive.

Proposition 2.4.2. *Given two polyhedra X and Y one has*

$$\mathsf{TC}(X \times Y) \leq \mathsf{TC}(X) + \mathsf{TC}(Y) - 1.$$

A proof of this proposition can be found in [15].

2.4.2 Lower bounds

An effective method to obtain a lower bound on $\mathsf{TC}(X)$ is given by studying cup products of certain classes in a cohomology ring of $X \times X$. This technique was introduced by Farber in [15] and later generalized in [21] by Farber and Grant, through the concept of weight of a cohomology class.

Definition 2.4.1. *Let X be a path-connected topological space and R a coefficient system on $X \times X$. A cohomology class $u \in H^*(X \times X; R)$ is said to have* ***weight*** *k if k is the largest integer such that for any open subset $A \subset X \times X$ with $\mathsf{TC}_X(A) \leq k$ one has $u|A = 0$; $u|A$ is the restriction of u to A. The weight of the zero cohomology class is defined to equal ∞.*

We will denote the weight of a cohomology class u by **wgt(u)**. The weight of a cohomology class depends on the coefficient system and this should be explicit when computations are made.

The proofs of the next three results can be found in [20].

Proposition 2.4.3. *If there exists a nonzero cohomology class $u \in H^*(X \times X; R)$ with $\text{wgt}(u) \geq k$, then $\mathsf{TC}(X) > k$.*

We will see in the next section that the above result often provides a better lower bound than the one given by Proposition 2.2.2.

The following two lemmas allow us to describe several cohomology classes with high weight.

Lemma 2.4.4. *For $u \in H^*(X \times X; R)$ one has $\text{wgt}(u) \geq 1$ if and only if*

$$u|\Delta_X = 0 \in H^*(X; R|\Delta_X),$$

where $u|\Delta_X$ denotes the restriction of u to $\Delta_X = \{(x,y) \in X \times X |\ x = y\}$; the diagonal of $X \times X$.

The classes which satisfy the condition $u|_{\Delta_X} = 0$ are usually called *zero divisors.*

Lemma 2.4.5. *Let $u \in H^n(X \times X; R)$ and $v \in H^m(X \times X; R')$ be two cohomology classes and denote by $uv \in H^{n+m}(X \times X; R \otimes R')$ their cup product. Then*

$$wgt(uv) \geq wgt(u) + wgt(v).$$

Remark 2.4.1. We observe that if R is an abelian group then any cohomology class $u \in H^*(X; R)$ induces a zero-divisor

$$\bar{u} = 1 \times u - u \times 1 \in H^*(X \times X; R)$$

since the definition of cup product implies $\bar{u}|\Delta_X = 1 \cup u - u \cup 1 = 0.$

Let $G = \mathbb{K}$ be a field. Then the cohomology ring of X allows us to easily find all the zero divisors. Through Künneth theorem

$$H^*(X \times X; \mathbb{K}) \cong H^*(X; \mathbb{K}) \otimes H^*(X; \mathbb{K}).$$

The zero divisors ideal is the kernel of the cup product homomorphism

$$\cup : H^*(X; \mathbb{K}) \otimes H^*(X; \mathbb{K}) \to H^*(X; \mathbb{K})$$

and a cohomology class $u \in H^n(X \times X; \mathbb{K})$ of the form

$$u = \sum_i a_i \times b_i, \quad a_i \in H^*(X; \mathbb{K}), \quad b_i \in H^{n-*}(X; \mathbb{K}),$$

is a zero divisor precisely when

$$\cup(u) = \sum_i a_i b_i = 0.$$

The tensor product $H^*(X; \mathbb{K}) \otimes H^*(X; \mathbb{K})$ is a graded $\mathbb{K}$-algebra with multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1||u_2|} u_1 u_2 \otimes v_1 v_2,$$

where $|v1|$ and $|u2|$ denote the degrees of the corresponding cohomology classes.

As an illustration we improve the result on the topological complexity of spheres mentioned in Example 2.2.3.

Proposition 2.4.6.

$$\mathsf{TC}(S^n) = \begin{cases} 2, & \textit{if } n \textit{ is odd,} \\ 3, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. In view of the arguments of Example 2.2.3, we will prove that $\mathsf{TC}(S^n) > 2$, for n even. Denote by $u \in H^n(S^n; \mathbb{Q})$ the fundamental class and by $1 \in H^0(S^n)$ the unit. Then $\theta = 1 \otimes u - u \otimes 1$ is a zero divisor. If n is even

$$\theta^2 = ((-1)^{n-1} - 1) \cdot u \otimes u = -2u \otimes u \neq 0.$$

Hence θ^2 is a nonzero class such that $\mathrm{wgt}(\theta^2) \geq 2$. In particular, by Proposition 2.4.3 we have that $\mathsf{TC}(S^n) > 2$, for n even. □

Theorem 2.4.1 and Proposition 2.4.3 fully determine the topological complexity of numerous other spaces as we will see in the next section.

2.5 Examples

We briefly describe known results about the topological complexity of several configuration spaces. A more detailed exposition with proofs can be found in Chapter 4 of [20].

The next two results can be proven by combining the upper bound (2.4) with the lower bound given by Proposition 2.4.3.

Proposition 2.5.1. *If Γ is a connected finite graph then*

$$\mathsf{TC}(\Gamma) = \begin{cases} 1, & \text{if } \Gamma \text{ is a tree,} \\ 2, & \text{if } \Gamma \text{ is homotopy equivalent to } S^1, \\ 3, & \text{otherwise.} \end{cases}$$

Proposition 2.5.2. *Denote by Σ_g the closed orientable surface of genus g. Then*

$$\mathsf{TC}(\Sigma_g) = \begin{cases} 3, & \text{if } g{=}0 \text{ or } g{=}1, \\ 5, & \text{if } g \geq 2. \end{cases}$$

The topological complexity of non-orientable surfaces is still an open problem. The inequality (2.4) implies that any surface S must have $\mathsf{TC}(S) \leq 5$. However, the method given by Proposition 2.4.3 apparently does not provide a lower bound greater than four, in the case of non-orientable surfaces. For example, the topological complexity of the Klein bottle K satisfies

$$4 \leq \mathsf{TC}(K) \leq 5$$

but the exact value is not known.

A curious result by M.Farber, S.Tabachnikov and S.Yuzvinsky [25] shows that the problem of determining the topological complexity of projective spaces is equivalent to solving their immersion problem, *i.e.* the problem of finding the minimal number k such that $\mathbb{R}P^n$ immerses in $\mathbb{R}^k$. Since the immersion problem for projective spaces is not fully solved, one may hope that topological complexity techniques give new insights to the immersion dimension problem for projective spaces. This perspective has been supported by M. Grant [32], J.González [29] and J. González-L. Zárate [31].

Theorem 2.5.3 ([25])**.** *For any $n \geq 1$ except $n = 1, 3, 7$,*

$$\mathsf{TC}(\mathbb{R}P^n) = \mathsf{I}(\mathbb{R}\mathsf{P}^\mathsf{n}) + 1$$

where $\mathsf{I}(\mathbb{R}\mathsf{P}^\mathsf{n})$ denotes the immersion dimension of the real projective space $\mathbb{R}P^n$, i.e., the smallest dimension k for which $\mathbb{R}P^n$ immerses in $\mathbb{R}^k$. Moreover for $n = 1, 3, 7$ one has $\mathsf{TC}(\mathbb{R}P^n) = n + 1$.

González [29] and Farber-Grant [21] studied, using different approaches, the topological complexity of lens spaces. The author of [29] gave estimations through certain equivariant maps between spheres, the *axial maps*. In [21] the authors obtained estimations using cohomological weights, a lower bound estimation method described in the previous section.

Theorem 2.5.4 ([29])**.** *Let L_m^{2n+1} be the lens space of dimension $2n+1$ with torsion m and assume that m divides $\binom{2n}{n}$. Then:*

1. *If m is even then $\mathsf{TC}(L_m^{2n+1}) \leq 4n$. Moreover, if m does not divide $\binom{2n-1}{n}$ one has $\mathsf{TC}(L_m^{2n+1}) = 4n$.*

2. *If m is odd and does not divide $\binom{2n-1}{n}$ then $\mathsf{TC}(L_m^{2n+1}) \geq 4n - 1$.*

Theorem 2.5.5 ([21])**.** *For any positive integers n and $m \geq 2$ one has*

$$\mathsf{TC}(L_m^{2n+1}) \leq 4n + 2.$$

Moreover, if m does not divide $\binom{2n}{n}$. Then one has

$$\mathsf{TC}(L_m^{2n+1}) = 4n + 2.$$

In Chapter 1 we introduced the configuration space $F(X, n)$ of n distinct points in X. Usually X is the Euclidean space $\mathbb{R}^m$ or a connected graph Γ. The next two theorems, by Farber-Yuzvinsky and Farber-Grant, describe completely the topological complexity of $F(X, n)$, when X is an Euclidean space.

Theorem 2.5.6 ([26])**.** *For any $n \geq 1$ one has*

$$\mathsf{TC}(F(\mathbb{R}^m, n)) = \begin{cases} 2n - 1, & \textit{for any odd } m, \\ 2n - 2, & \textit{for } m = 2. \end{cases}$$

Theorem 2.5.7 ([22])**.** *For any $n \geq 1$ and m even*

$$\mathsf{TC}(F(\mathbb{R}^m, n)) = 2n - 2.$$

Consider the configuration space $F(\Gamma, n)$, where Γ is a connected graph. Call a vertex *essential* if it has degree at least three.

Theorem 2.5.8 ([17])**.** *If Γ has at least an essential vertex, then*

$$\mathsf{TC}(F(\Gamma, n)) \leq 2m(\Gamma) + 1$$

where $m(\Gamma)$ is the number of essential vertices in Γ.

As we have seen in Proposition 2.2.1 the topological complexity of a space is an homotopy invariant. The homotopy type of an aspherical space X, *i.e.* $\pi_i(X) = 0$ for any $i > 1$, depends exclusively on the fundamental group of X; see [34], section 1.B. In Chapter 3 we will present some connections between algebraic properties of the fundamental group of a polyhedron X and the number $\mathsf{TC}(X)$.

Definition 2.5.1. *The **topological complexity of a group** G is defined as the topological complexity of the associated aspherical space with fundamental group G. Namely,*

$$\mathsf{TC}(G) = \mathsf{TC}(K(G,1)),$$

where $K(G,1)$ is an Eilenberg-MacLane space.

Definition 2.5.2. *To a finite graph Γ with vertex set V and edge set E we may associate a **right-angled Artin group** (RAAG) (also known as a **graph group**)*

$$G_\Gamma = |v \in V; vw = wv \quad \textit{iff} \quad (v,w) \in E|,$$

see [6], [42].

In the case when Γ is a complete graph G_Γ is a free abelian group of rank $n = |V|$; in the other extreme, when Γ has no edges the group G_Γ is the free group of rank n. In general G_Γ interpolates between the free and free abelian groups.

In [9], D. Cohen and G. Pruidze determined the topological complexity of right angled Artin groups in terms of the properties of the graph.

Theorem 2.5.9 ([9])**.** *Let Γ be a graph and G_Γ the respective right angled Artin group. Then*

$$\mathsf{TC}(G_\Gamma) = z(\Gamma) + 1,$$

where

$$z(\Gamma) = \max_{K_1, K_2} |K_1 \cup K_2|$$

is the maximal number of vertices that support two complete subgraphs in Γ.

In Chapter 5 we will study the topological complexity of right-angled Artin groups generated by random graphs.

2.6 Symmetric topological complexity

Symmetric motion planning is motion planning with the extra requirement that if a motion planner assigns a certain path connecting a point A to a point B then it must assign the same path (with reverse orientation) to connect B to A; see Definition 2.6.1 below.

In this section we present the symmetric version of the notion of topological complexity. The concept of Symmetric Topological Complexity was first introduced by Farber and Grant in [23]; we refer back to that paper for a deeper discussion. Gonzalez and Landweber [30] applied this concept to the study of projective and lens spaces and obtained a surprising relation between the symmetric topological complexity of projective spaces and their embedding dimension. Unlike the (non-symmetric) topological complexity of a space, the symmetric topological complexity is not an homotopy invariant.

Definition 2.6.1. *Let* $\mathfrak{p} : PX \to X \times X$ *be the path-fibration described in (2.1). A* ***symmetric motion planner*** *in* X *is a (possibly discontinuous) map*

$$s : X \times X \to PX$$

such that $\mathfrak{p} \circ s = Id_{X \times X}$ *and for every* $t \in [0,1]$ *one has*

$$s(A,A)(t) = A \quad \text{and} \quad s(A,B)(t) = s(B,A)(1-t), \quad A, B \in X.$$

The path fibration

$$\mathfrak{p} : PX \to X \times X$$

can be restricted to a fibration

$$\mathfrak{p}' : P'X \to F(X;2) \tag{2.5}$$

where

$$F(X,2) = \{(x,y) \in X \times X |\ x \neq y\}$$

is the space defined earlier in Example 1.1.5 and $P'X$ is the subspace

$$\{\gamma : [0,1] \to X |\ \gamma(0) \neq \gamma(1)\} \subset PX$$

of paths with distinct endpoints. Both spaces $P'X$ and $F(X,2)$ admit free $\mathbb{Z}_2$-actions defined by path reversing and factors interchange, respectively. Besides $\mathfrak{p}' : P'X \to F(X,2)$ is an equivariant map of free $\mathbb{Z}_2$-spaces and induces a fibration

$$\mathfrak{p}_S := \mathfrak{p}'/\mathbb{Z}_2 : P'X/\mathbb{Z}_2 \to B(X,2). \tag{2.6}$$

Here $B(X,2) = F(X,2)/\mathbb{Z}_2$ is the space previously defined in Example 1.1.5, namely the space of unordered pairs of distinct points in X.

Definition 2.6.2. *The* ***Symmetric Topological Complexity*** *of* X*, denoted by* $\mathsf{TC}^S(X)$*, is the number*

$$\mathsf{TC}^S(X) = 1 + genus(\mathfrak{p}_S), \tag{2.7}$$

where $genus(\mathfrak{p}_S)$ *is the Schwarz genus of the fibration* $\mathfrak{p}_S$*.*

By definition, the number $genus(\mathfrak{p}_S)$ is the minimal integer k such that there is an open cover $\overline{U}_1, \ldots, \overline{U}_k$ of $B(X,2)$ with the property that for every i there is a continuous section $\overline{s}_i : \overline{U}_i \to P'X/\mathbb{Z}_2$ of the fibration $\mathfrak{p}_S$. Here by "open" we mean an open set for the quotient topology carried by $B(X,2)$. Moreover, there is a cover $\{U_1, \ldots, U_k\}$ of $F(X,2)$ such that, for every i, $U_i/\mathbb{Z}_2 = \overline{U}_i$ and there is an equivariant lift $s_i : U_i \to P'X$ of the local section $\overline{s}_i$. Hence a local section of the fibration p_S induces a symmetric local section of the path-fibration $\mathfrak{p} : PX \to X \times X$. In particular, since $\mathsf{TC}_X(\Delta_X) = 1$ and, by Remark 2.3.1, it holds that

$$\mathsf{TC}(X) \leq \mathsf{TC}_X(\Delta_X) + \mathsf{TC}_X((X \times X)\backslash\Delta_X),$$

one has

$$\mathsf{TC}(X) \leq \mathsf{TC}^S(X). \tag{2.8}$$

Inequality (2.8) can be an equality. For any positive integer n it holds

$$\mathsf{TC}(S^{2n}) = \mathsf{TC}^S(S^{2n}) = 3;$$

see [23]. Another example is given by the complex projective space $\mathbb{C}P^n$. Namely, one has

$$\mathsf{TC}(\mathbb{C}P^n) = 2n + 1 = \mathsf{TC}^S(\mathbb{C}P^n).$$

The non-symmetric side of the equality was established in [25] and the symmetric side was proven in [30].

One has an universal upper bound

$$\mathsf{TC}^S(X) \leq 2\dim X + 2;$$

this should be compared with (2.4). The above inequality is derived from the fact that, for any fibration $p : E \to B$, one has $genus(p) \leq \dim B + 1$. In our case

$p = \mathfrak{p}_S : P'X/\mathbb{Z}_2 \to B(X,2)$ and $\dim B(X,2) = 2\dim X$. Hence one has the inequality $\mathsf{TC}^S(X) = 1 + g(\mathfrak{p}_S) \leq 2\dim X + 2$.

Recall from Example 1.1.5 that there are homotopy equivalent spaces X_1 and X_2 such that $F(X_1, n)$ is not homotopy equivalent to $F(X_2, n)$. This implies that, unlike the non-symmetric version, the number $\mathsf{TC}^S(X)$ is not an homotopy invariant.

Chapter 3

Topological Complexity of Spaces with Small Fundamental Group

This chapter is an exposition of joint work with M. Farber [7]. There we established sharp upper bounds for the topological complexity $\mathsf{TC}(X)$, where X is a polyhedron such that $\pi_1(X)$ is "small"; either $\pi_1(X)$ is cyclic of order ≤ 3 or "small" cohomological dimension.

3.1 Introduction

Let X be a path-connected polyhedron. We have seen in Theorem 2.4.1 is that $\mathsf{TC}(X)$ admits the upper bound (2.4). Namely,

$$\mathsf{TC}(X) \leq 2\dim X + 1. \tag{3.1}$$

Examples when the above inequality is sharp include orientable surfaces of genus greater than one or the connected sum of two n-torus.

For simply connected spaces, Theorem 2.4.1 gives the stronger upper bound

$$\mathsf{TC}(X) \leq \dim(X) + 1, \tag{3.2}$$

which is sharp for example when X is a simply connected closed symplectic manifold X, see [25].

A natural question is if (3.1) can be improved under assumptions on the fundamental group $\pi_1(X)$. The assumption that $\pi_1(X) = \mathbb{Z}_2$ leads to the theorem below, which will be proved later in this chapter.

Theorem 3.1.1. *Let X be a cell complex with $\pi_1(X) = \mathbb{Z}_2$. Then*

$$\mathsf{TC}(X) \leq 2\dim(X). \tag{3.3}$$

Furthermore, for a closed manifold X with $\pi_1(X) = \mathbb{Z}_2$ it holds that

$$\mathsf{TC}(X) \leq 2\dim(X) - 1 \tag{3.4}$$

assuming that $w^n = 0$, where $n = \dim(X)$ and $w \in H^1(X; \mathbb{Z}_2)$ is the generator.

Notice that also (3.3) is sharp since $\mathsf{TC}(\mathbb{R}P^n) = 2n$ when n is a power of 2; see Corollary 8.2 of [25].

Theorem 3.1.1 should be compared with Theorem 3.5 of [3] given below:

Theorem 3.1.2 ([3])**.** *For a closed connected n-dimensional manifold X with $\pi_1(X) = \mathbb{Z}_2$ one has $\mathsf{cat}(X) = \dim(X) + 1$ if and only if $w^n \neq 0 \in H^n(X; \mathbb{Z}_2)$ where $w \in H^1(X; \mathbb{Z}_2)$ is the generator.*

We now have a clear picture of the case when the space X has fundamental group $\pi_1(X) = \mathbb{Z}_2$. Theorem 3.1.3 addresses the case when $\pi_1(X) = \mathbb{Z}_3$ and will be proved later in this chapter.

Theorem 3.1.3. *Let X be a finite cell complex such that $\pi_1(X) = \mathbb{Z}_3$.*

1. *Assume that either $\dim X$ is odd or $\dim X = 2n$ is even and the 3-adic expansion of n contains at least one digit 2. Then,*

$$\mathsf{TC}(X) \leq 2\dim(X). \tag{3.5}$$

2. *For any integer $n \geq 1$ having only the digits 0 and 1 in its 3-adic expansion there exists a finite polyhedron X of dimension $2n$ with $\pi_1(X) = \mathbb{Z}_3$ and*

$$\mathsf{TC}(X) = 2\dim(X) + 1.$$

If X is the lens space L_3^{2n+1}, for an n that has only the digits 0 and 1 in its 3-adic expansion, one has $\mathsf{TC}(X) = 2\dim(X)$; the assumption on n is equivalent to the assumption in Theorem 2.5.5 with $m = 3$. This shows that the inequality (3.5) is sharp.

We now look at the case where the fundamental group has finite cohomological dimension. The following theorem is an adaptation of Dranishnikov result relating the Lusternik-Schnirelmann category of a space and its fundamental group.

Theorem 3.1.4. *Let X be a finite cell complex. Then one has*

$$\mathsf{TC}(X) \leq \begin{cases} \dim(X) + 2\mathrm{cd}(\pi_1(X)), & \textit{if } \dim(X) \textit{ is odd,} \\ \\ \dim(X) + 2\mathrm{cd}(\pi_1(X)) + 1, & \textit{if } \dim(X) \textit{ is even.} \end{cases} \tag{3.6}$$

Proof. This theorem is a consequence of a recent theorem of Dranishnikov [11] regarding Lusternik - Schnirelmann category. The theorem states that for a cell complex X with fundamental group $\pi_1(X)$ of finite cohomological dimension one has

$$\mathsf{cat}(X) \leq \left\lceil \frac{\dim(X) - 1}{2} \right\rceil + \mathrm{cd}(\pi_1(X)) + 1. \tag{3.7}$$

Inequality (3.6) follows from (3.7) and from the inequality

$$\mathsf{TC}(X) \leq 2 \cdot \mathsf{cat}(X) - 1,$$

see [15]. □

Remark 3.1.1. Notice that 3.1.4 improves (3.1) whenever the cohomological dimension of the fundamental group of X is smaller than half of the dimension of X. Theorem 3.1.4 does not improve (3.1) when $\dim X = 2$. If $\dim X = 2$ and $\mathrm{cd}(\pi_1(X)) = 1$, *i.e.* $\pi_1(X)$ is free, then X is homotopy equivalent to a wedge of circles and 2-spheres. Then either $\mathsf{TC}(X) = 2$ (exactly when $X \sim S^1$) or $\mathsf{TC}(X) = 3$.

The topological complexity of spaces with free abelian fundamental group is also described by the following result:

Theorem 3.1.5 ([9])**.** *Let X be the l-skeleton of the n-torus, $n \geq l \geq 2$. Then $\mathsf{TC}(X) = \min\{n+1, 2l+1\}$.*

3.2 Necessary and sufficient conditions for $\mathsf{TC}(X) \leq 2\dim X$

We will show in this section that the upper bound

$$\mathsf{TC}(X) \leq 2\dim X$$

is equivalent to the vanishing of a power of a certain cohomology class; the primary obstruction to the existence of a continuous section of the path fibration

$$\mathfrak{p} : PX \to X \times X$$

defined in (2.1).

Let $x_0 \in X$ be the base point of X. The fibre $F = \mathfrak{p}^{-1}(x_0, x_0)$ is the space ΩX of all loops in X based at x_0. Clearly F is disconnected; the set of path-components is in bijection with $\pi_1(X, x_0)$. The primary "homological obstruction" to the existence of a continuous section of $\mathfrak{p}$ is a cohomology class

$$\theta \in H^1(X \times X, \{\widetilde{H}_0(F)\}), \tag{3.8}$$

where $\{\widetilde{H}_0(F)\}$ denotes a local coefficient system over $X \times X$ which we will describe later.

Denote by $\mathfrak{p}_{2n} : P_{2n}X \to X \times X$ the $2n$-fold fiberwise join of the path space fibration $\mathfrak{p} : PX \to X \times X$ defined in (2.1); for details on this construction (also called the sum of the fibration) consult [45], Chapter 2. The fibre F_{2n} of $\mathfrak{p}_{2n}$ is the $2n$-fold join $\Omega X * \ldots * \Omega X$. It follows by the properties of the join of spaces that F_{2n} is $(2n-2)$-connected and thus, by Hurewicz Theorem,

$$\{\pi_{2n-1}(F_{2n})\} = \{H_{2n-1}(F_{2n})\}$$

and the primary obstruction

$$\theta_{2n} \in H^{2n}(X \times X; \{H_{2n-1}(F_{2n})\})$$

to the existence of a continuous section of $\mathfrak{p}_{2n}$ lies in the top cohomology group of $X \times X$ and therefore θ_{2n} is the only obstruction to the existence of a section of $\mathfrak{p}_{2n}$.

In [45] Schwarz showed that is possible to reduce the computation of the genus of a fibration to the study of the existence of a section in a certain join of that fibration.

Theorem 3.2.1 ([45]). *Let $p : E \to B$ be a fibration. Then genus$(p) \leq k$ if and only if p_k has a continuous section, where p_k is the k-fold fiberwise join of p.*

The relation between θ and θ_{2n}, respectively the primary obstructions associated to the fibrations $\mathfrak{p}$ and $\mathfrak{p}_{2n}$, is given by Theorem 2 of [45], which establishes that

$$\{H_{2n-1}(F_{2n})\} = \otimes_{i=1}^{2n}\{\widetilde{H}_0(F)\}$$

and that θ_{2n} is the $2n$-fold cup product of θ, *i.e.*

$$\theta_{2n} = \theta^{2n}. \tag{3.9}$$

By the (3.9) and Theorem 3.2.1 one has the following result:

Corollary 3.2.2. $\mathsf{TC}(X) \leq 2n$ *if and only if* $\theta^{2n} = 0$.

The next step is to describe explicitly the primary obstruction to the path fibration $\mathfrak{p}$.

Denote the fundamental group of X by $G = \pi_1(X, x_0)$ and the kernel of the associated augmentation homomorphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ by $I = \ker(\epsilon) \subset \mathbb{Z}[G]$. An element of I is a finite sum of the form $\sum n_i g_i$ such that $\sum n_i = 0$, where $n_i \in \mathbb{Z}$ and $g_i \in G$. One can view I and $\mathbb{Z}[G]$ as left $\mathbb{Z}[G \times G]$-modules via the action

$$(g, h) \cdot \sum n_i g_i = \sum n_i (g g_i h^{-1}), \quad g, h \in G. \tag{3.10}$$

Since I and $\mathbb{Z}[G]$ are $\mathbb{Z}[\pi_1(X \times X)]$-left modules they determine local coefficient systems over $X \times X$; consult [48], Chapter 6.

By Theorem 3.3 (Chapter 6) of [48], crossed homomorphisms determine one-dimensional cohomology classes. A *crossed homomorphism* is a map $f : G \times G \to I$ that satisfies the identity

$$f((g, h)(g', h')) = f(g, h) + (g, h) \cdot f(g'h')$$

where $g, g', h, h' \in G$.

Fix the crossed homomorphism $f : G \times G \to I$ given by

$$f(g,h) = gh^{-1} - 1 \tag{3.11}$$

and denote the corresponding one-dimensional cohomology class by

$$\mathfrak{v} \in H^1(X \times X; I). \tag{3.12}$$

As groups $\mathbb{Z}[G]$ and $H_0(F)$ are identical since they are free abelian groups on the same number of generators. Hence

$$I = \widetilde{H}_0(F),$$

as groups. We will show that in fact I and $\{\widetilde{H}_0(F)\}$ can be identified as local coefficient systems and the class $\mathfrak{v}$ represents exactly the primary homological obstruction θ. This leads to the following theorem.

Theorem 3.2.3. *Let X be a cell complex of dimension $n = \dim(X) \geq 2$. One has*

$$\mathsf{TC}(X) \leq 2n$$

if and only if the $2n$-th power

$$\mathfrak{v}^{2n} = 0 \in H^{2n}(X \times X; I^{2n})$$

vanishes. Here $I^{2n} = I \otimes_{\mathbb{Z}} I \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} I$ denotes the tensor product over $\mathbb{Z}$ of $2n$ copies of I, equipped with the diagonal action of $G \times G$, and $\mathfrak{v}^{2n}$ is the cup-product $\mathfrak{v} \cup \mathfrak{v} \cup \cdots \cup \mathfrak{v}$ of $2n$ copies of $\mathfrak{v}$, the class described in (3.12).

Proof. Fix $(x_0, x_0) \in X \times X$ and the corresponding fibre $F = F_{(x_0,x_0)} = \mathfrak{p}^{-1}((x_0, x_0)) = \Omega X$. To a loop $\sigma : [0,1] \to X \times X$, where $\sigma(t) = (\alpha(t), \beta(t))$, with $\sigma(0) = \sigma(1) = (x_0, x_0)$, we may associate an homotopy

$$\Sigma : \mathfrak{p}^{-1}(x_0, x_0) \times [0,1] \to PX$$

using the homotopy lifting property of the fibration $\mathfrak{p}$.

Let Σ be defined by the formula

$$\Sigma(\omega,\tau)(t) = \begin{cases} \alpha(3t+\tau), & \text{for } 0 \leq t \leq \frac{1-\tau}{3}, \\ \omega(\frac{3t+\tau-1}{1+2\tau}), & \text{for } \frac{1-\tau}{3} \leq t \leq \frac{2+\tau}{3}, \\ \beta(-3t+\tau+3), & \text{for } \frac{2+\tau}{3} \leq t \leq 1, \end{cases}$$

Σ satisfies the identities $\Sigma(\omega,1)=\omega$ and $\mathfrak{p}(\Sigma(\omega,\bullet))=\sigma$, and induces a map

$$F_{\sigma(1)} \xrightarrow{\rho} F_{\sigma(0)}$$

defined by $\omega \mapsto \Sigma(\omega,0)$. The monodromy action of σ on ΩX can be simply described by

$$\omega \mapsto \Sigma(\omega,0) = \alpha\omega\bar{\beta}; \tag{3.13}$$

where $\bar{\beta}$ represents the inverse loop of β and $(\alpha,\beta)=\sigma$.

The induced map ρ_* on homology

$$\rho_* : \widetilde{H}_0(F_{\sigma(1)}) \to \widetilde{H}_0(F_{\sigma(0)})$$

is an isomorphism and the above construction gives a monodromy action on $\widetilde{H}_0(F)$ that is identical to the action on I presented in (3.10). Hence

$$I = \{\widetilde{H}_0(F)\}$$

as local coefficient systems.

Using Corollary 3.2.2 we complete the proof by identifying $\mathfrak{v}$ with the primary homological obstruction $\theta \in H^1(X \times X, I)$.

Assume that X has a single 0-cell x_0 (quotient by a maximal tree in the 1-skeleton if necessary) and denote by ω_0 the corresponding constant loop. The homological obstruction θ associates with any oriented 1-cell of $X \times X$ the formal difference, in $\widetilde{H}_0(\Omega X) = I$, between the connected components of $\Sigma(\omega_0, 0)$ and ω_0, where Σ is induced by σ (a loop representing the 1-cell), by the construction described before. Given an oriented 1-cell e of X consider the 1-cells $e \times x_0$ and $x_0 \times e$ in $X \times X$ and let g to be the loop in X representing e. By formula (3.13) the crossed homomorphism $f' : G \times G \to I$ representing θ is given by

$$f'(g,1) = g - 1, \quad f'(1,h) = h^{-1} - 1, \quad h \in G.$$

Using the definition of crossed homomorphism it follows that

$$f'(g,h) = f'((g,1)(1,h)) = f'(g,1) + (g,1)f'(1,h) = gh^{-1} - 1 = f(g,h).$$

Thus $\theta = \mathfrak{v}$. $\square$

Corollary 3.2.4. *Let X be a cell complex with $\mathsf{TC}(X) = 2\dim(X) + 1$. Then the topological complexity of the Eilenberg - MacLane complex $Y = K(\pi_1(X), 1)$ satisfies*

$$\mathsf{TC}(Y) \geq 2\dim(X) + 1.$$

Proof. If X $\dim X = 1$ then X is aspherical and the statement above is trivial. Hence we may assume that $n = \dim(X) \geq 2$ and apply Theorem 3.2.3.

Consider local systems I_X on $X \times X$ and I_Y on $Y \times Y$ and cohomology classes $\mathfrak{v}_X \in H^1(X \times X; I_X)$ and $\mathfrak{v}_Y \in H^1(Y \times Y; I_Y)$ defined as in (3.12). The canonical map $f : X \to Y$ inducing an isomorphism of fundamental groups satisfies

$$(f \times f)^*(I_Y) = I_X \text{ and } (f \times f)^*(\mathfrak{v}_Y) = \mathfrak{v}_X.$$

By Theorem 3.2.3, we obtain $(\mathfrak{v}_X)^{2n} \neq 0$. On the other hand, if $(\mathfrak{v}_X)^{2n} \neq 0$ then $(\mathfrak{v}_Y)^{2n} \neq 0$. Inequality $\mathsf{TC}(Y) \geq 2n + 1$ now follows from Proposition 2.4.3 since $\mathrm{wgt}(\mathfrak{v}_Y) \geq 1$; see Lemma 3.2.5. $\square$

The proof of theorems 3.1.1 and 3.1.3 are based on Theorem 3.2.3 combined with the results below.

Lemma 3.2.5. *The restriction of the class $\mathfrak{v} = \mathfrak{v}_X$ to the diagonal $\Delta_X \subset X \times X$ vanishes, i.e.,*

$$\mathfrak{v}_X|\Delta_X = 0 \in H^1(X; I|X). \tag{3.14}$$

In particular, $\mathrm{wgt}(\mathfrak{v}_X) \geq 1$*; see Lemma 2.4.4.*

Proof. Just notice that the crossed homomorphism induced by f, given by (3.11), is trivial when restricted to the diagonal $\Delta_G \subset G \times G$, *i.e.* $f(g, g) = 0$ for all $g \in G$. $\square$

Note that the local system $I|X$ corresponds to the ideal I viewed with the left G-action

$$g \cdot \sum n_i g_i = \sum n_i \cdot (g g_i g^{-1}),$$

where $g, g_i \in G$ and $\sum n_i = 0$.

The class $\mathfrak{v} = \mathfrak{v}_x$ can be described as follows:

Lemma 3.2.6. *One has*

$$\mathfrak{v}_X = \beta(1) \in H^1(X \times X; I)$$

where

$$\beta : H^0(X \times X; \mathbb{Z}) \to H^1(X \times X; I)$$

is Bockstein homomorphism corresponding to the exact sequence of left $\mathbb{Z}[G \times G]$-modules

$$0 \to I \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Proof. Cohomology with local coefficients of a space can be seen as the equivariant cohomology of its universal cover, see [34]. Let $\widetilde{X}$ denote the universal cover of X and let $\widetilde{x}_0 \in \widetilde{X}$ be a lift of the base point $x_0 \in X$. Consider the singular chain complex $S_* = S_*(\widetilde{X} \times \widetilde{X})$. This is a free left $\mathbb{Z}[G \times G]$-module. The equivariant cohomology $E^*_{G\times G}(\widetilde{X} \times \widetilde{X}, \mathbb{Z}[G])$ is generated by $\mathbb{Z}[G \times G]$-homomorphisms from S_* to $\mathbb{Z}[G]$. We may identify $S_0(\widetilde{X} \times \widetilde{X})$ with the free abelian group generated by the points of $\widetilde{X} \times \widetilde{X}$. Consider a $\mathbb{Z}[G \times G]$-homomorphism

$$k : S_0(\widetilde{X} \times \widetilde{X}) \to \mathbb{Z}[G]$$

associating an element of G with every point of $\widetilde{X} \times \widetilde{X}$ and such that $k(\widetilde{x}_0, \widetilde{x}_0) = 1 \in G$. Hence $k(g\widetilde{x}_0, h\widetilde{x}_0) = gh^{-1}$ for $g, h \in G$. The cochain $\epsilon \circ k : S_0 \to \mathbb{Z}$ represents the class $1 \in H^0(X \times X; \mathbb{Z})$ and the Bockstein image $\beta(1) \in H^1(X \times X; I)$ is represented by the composition

$$\delta(k) : \; S_1(\widetilde{X} \times \widetilde{X}) \xrightarrow{\partial} S_0(\widetilde{X} \times \widetilde{X}) \xrightarrow{k} I$$

taking values in $I \subset \mathbb{Z}[G]$ (as follows from the definition of the Bockstein homomorphism). A crossed homomorphism $f' : G \times G \to I$ associated to $\beta(1)$ can be found as follows, see [48], Chapter 6, §3. Given a pair $(g, h) \in G \times G = \pi_1(X \times X, (x_0, x_0))$, realize it by a loop $\sigma : ([0,1], \partial[0,1]) \to (X \times X, (x_0, x_0))$, then lift σ to the covering $\widetilde{\sigma} : ([0,1], 0) \to (\widetilde{X} \times \widetilde{X}, (\widetilde{x}_0, \widetilde{x}_0))$ and apply the cocycle $\delta(k)$ to $\widetilde{\sigma}$, viewed as a singular 1-simplex in $\widetilde{X} \times \widetilde{X}$. We obtain

$$f'(g, h) = k(g\widetilde{x}_0, h\widetilde{x}_0) - k(\widetilde{x}_0, \widetilde{x}_0) = gh^{-1} - 1$$

for all $g, h \in G$. This coincides with the crossed homomorphism describing $\mathfrak{v}_X$, see (3.11). Thus $\beta(1) = \mathfrak{v}_X$. $\square$

Corollary 3.2.7. *The order of the class $\mathfrak{v}_X \in H^1(X \times X; I)$ equals the cardinality $|G|$ of the fundamental group of X. In particular $\mathfrak{v}_X = 0$ if and only if X is simply connected.*

Proof. Consider the exact sequence

$$H^0(X \times X; I) \to H^0(X \times X; \mathbb{Z}[G]) \xrightarrow{\epsilon} H^0(X \times X; \mathbb{Z}) \xrightarrow{\beta} H^1(X \times X; I).$$

Note that $H^0(X \times X; \mathbb{Z}[G])$ is isomorphic to the set of elements $a = \sum n_i g_i \in \mathbb{Z}[G]$ which are invariant with respect to $G \times G$-action, see [48], Chapter 6, Theorem 3.2.

If G is infinite then $H^0(X \times X; \mathbb{Z}[G]) = 0$ as there are no invariant elements in the group ring. Since $H^0(X \times X; \mathbb{Z}) = \mathbb{Z}$ this implies that in this case the class $\mathfrak{v}_X \in H^1(X \times X; I)$ generates an infinite cyclic subgroup.

In the case when G is finite any $G \times G$-invariant element of $\mathbb{Z}[G]$ is a multiple of $N = \sum_{g \in G} g$ and $H^0(X \times X; I) = 0$. Hence the group $H^0(X \times X; \mathbb{Z}[G])$ is infinite cyclic generated by N and since $\epsilon(N) = |G|$, the exact sequence

$$0 \to H^0(X \times X; \mathbb{Z}[G]) \xrightarrow{\epsilon} H^0(X \times X; \mathbb{Z}) \xrightarrow{\beta} H^1(X \times X; I)$$

becomes

$$0 \to \mathbb{Z} \xrightarrow{\times |G|} \mathbb{Z} \xrightarrow{\beta} H^1(X \times X; I).$$

It follows that the subgroup of $H^1(X \times X; I)$ generated by the class $\mathfrak{v}_X$ is cyclic of order $|G|$. $\square$

We can now prove Theorems 3.1.1 and 3.1.3.

3.3 Proof of Theorem 3.1.1

Let X be a connected cell complex with $\pi_1(X) = \mathbb{Z}_2 = G$. Then the augmentation ideal $I = \ker[\epsilon : \mathbb{Z}[G] \to \mathbb{Z}]$ is isomorphic to $\mathbb{Z}$ as an abelian group but not as local systems on $X \times X$. Denote by $g \in G$ the unique nontrivial element of G. Then both the classes $(g, 1), (1, g) \in G \times G$ act as multiplication by -1 on $\mathbb{Z} = I$. Thus I is the local system of "twisted integers". It follows that the tensor square $I \otimes_{\mathbb{Z}} I$ is a trivial local coefficient system isomorphic to $\mathbb{Z}$. Note that Theorem 3.2.3 is applicable since we must have $n = \dim(X) \geq 2$.

Consider the canonical class $\mathfrak{v} = \mathfrak{v}_X \in H^1(X \times X; I)$ and its square

$$\mathfrak{v}^2 \in H^2(X \times X; \mathbb{Z}).$$

Since $H^1(X; \mathbb{Z}) = 0$ the Künneth theorem gives

$$H^2(X \times X) = H^2(X) \otimes H^0(X) \oplus H^0(X) \otimes H^2(X);$$

all omitted coefficients are identically $\mathbb{Z}$. Hence we may write

$$\mathfrak{v}^2 = a \times 1 + 1 \times b, \quad a, b \in H^2(X; \mathbb{Z}).$$

By Lemma 3.2.5 one has $a + b = 0$, and by Corollary 3.2.7 both classes a and b are of order two: $2a = 0 = 2b$. Thus we may rewrite

$$\mathfrak{v}^2 = a \times 1 + 1 \times a$$

and

$$\mathfrak{v}^{2n} = (\mathfrak{v}^2)^n = (a \times 1 + 1 \times a)^n = \sum_{i=0}^{n} \binom{n}{i} a^i \times a^{n-i}.$$

If n is odd then for any i either $a^i = 0$ or $a^{n-i} = 0$ for dimensional reasons. Thus $\mathfrak{v}^{2n} = 0$. If n is even then

$$\mathfrak{v}^{2n} = \binom{n}{n/2} a^{n/2} \times a^{n/2} = 0$$

since the binomial coefficient $\binom{n}{n/2}$ is always even and $2a = 0$. By Theorem 3.2.3 we obtain $\mathsf{TC}(X) \leq 2n$.

Assume now that X is a closed manifold satisfying $\pi_1(X) = \mathbb{Z}_2$ and $w^n = 0$ where $w \in H^1(X;\mathbb{Z}_2)$ is the generator. By the theorem of Berstein mentioned earlier (Theorem 3.1.2) one has $\mathsf{cat}(X) \leq \dim(X)$. The statement (3.4) follows now from the inequality $\mathsf{TC}(X) \leq 2\mathsf{cat}(X) - 1$, see [15]. This proves the second statement of the theorem and thus completes the proof.

3.4 Proof of Theorem 3.1.3

Let X be a connected cell complex with fundamental group $\pi_1(X) = G = \mathbb{Z}_3$. We represent G as the multiplicative group $\{1, t, t^2\}$ holding the identity $t^3 = 1$. The group ring $\mathbb{Z}[G]$ is the ring of polynomials with integer coefficients of the form $a+bt+ct^2$, with the usual operations and the extra relation $t^3 = 1$. The augmentation ideal I has rank 2 with generators $\alpha = t - 1$ and $\beta = t^2 - t$. As a $\mathbb{Z}[G \times G]$-module, I is defined by

$$(t, 1) \cdot \alpha = \beta, \quad (t, 1) \cdot \beta = -\alpha - \beta,$$

and

$$(1, t) \cdot \alpha = -\alpha - \beta, \quad (1, t) \cdot \beta = \alpha.$$

Consider the canonical class $\mathfrak{v}_X \in H^1(X \times X; I)$ and respective square

$$\mathfrak{v}_X^2 \in H^2(X \times X; I \otimes I).$$

The local system $I \otimes I$ has rank 4 and is generated by the elements $\alpha \otimes \alpha$, $\alpha \otimes \beta$, $\beta \otimes \alpha$ and $\beta \otimes \beta$ with $G \times G$ acting diagonally; for example

$$\begin{aligned} (t, 1) \cdot \alpha \otimes \alpha &= \beta \otimes \beta, \\ (t, 1) \cdot \alpha \otimes \beta &= \beta \otimes (-\alpha - \beta) = -\beta \otimes \alpha - \beta \otimes \beta. \end{aligned}$$

and so forth.

Consider the homomorphism

$$T : I \otimes I \to I \otimes I$$

which interchanges the factors. One has $T(\alpha \otimes \beta) = \beta \otimes \alpha$, $T(\beta \otimes \alpha) = \alpha \otimes \beta$ and T acts trivially on the two other generators $\alpha \otimes \alpha$ and $\beta \otimes \beta$. It is easy to verify that the diagonal action commutes with the interchange of factors. Hence T is a $\mathbb{Z}[G \times G]$-homomorphism and therefore a homomorphism of local systems.

Let $I \wedge I \subset I \otimes I$ denote the subgroup generated by the element $\alpha \otimes \beta - \beta \otimes \alpha$. It is easily verifiable that $I \wedge I = \mathbb{Z}$ has a trivial $\mathbb{Z}[G \times G]$-action; in particular it is a $\mathbb{Z}[G \times G]$-submodule of $I \otimes I$. Denote the factor module by $S(I)$; it is the symmetric square of I. We have the following exact sequence of local systems

$$0 \to I \wedge I \xrightarrow{i} I \otimes I \xrightarrow{j} S(I) \to 0$$

(recall that $I \wedge I = \mathbb{Z}$ is trivial) which induces an exact sequence

$$\ldots \to H^n(X \times X; I \wedge I) \xrightarrow{i_*} H^n(X \times X; I \otimes I) \xrightarrow{j_*} H^n(X \times X; S(I)) \to \ldots \quad (3.15)$$

The skew-commutativity property of cup-products implies that $T_*(\mathfrak{v}_X^2) = -\mathfrak{v}_X^2$. Since $j = j \circ T$ we obtain $j_*(\mathfrak{v}_X^2) = j_* T_*(\mathfrak{v}_X^2) = -j_*(\mathfrak{v}_X^2)$; thus $2j_*(\mathfrak{v}_X^2) = 0$. On the other hand, by Corollary 3.2.7 one has $3j_*(\mathfrak{v}_X^2) = 0$. Hence

$$j_*(\mathfrak{v}_X^2) = 0 \in H^2(X \times X; S(I)).$$

From the long exact sequence (3.15) we obtain

$$\mathfrak{v}_X^2 = i_*(w) \quad (3.16)$$

for some $w \in H^2(X \times X; \mathbb{Z})$.

Let $A : I \otimes I \to I \wedge I = \mathbb{Z}$ be the map given by $A(x) = x - T(x)$ for $x \in I \otimes I$. Clearly A is a homomorphism of local systems and $A \circ i : I \wedge I \to I \wedge I$ is multiplication by 2. Hence we obtain $2w = A_* \circ i_*(w) = A_*(\mathfrak{v}_X^2)$ which implies

$$6w = 0$$

since $3\mathfrak{v}_X = 0$.

Applying Künneth theorem with respect to $H^2(X \times X; \mathbb{Z})$ and using the fact that $H^1(X; \mathbb{Z}) = 0$ one can write

$$w = a \times 1 + 1 \times b$$

where $a, b \in H^2(X; \mathbb{Z})$ with $6a = 0 = 6b$. Then

$$\mathfrak{v}_X^{2n} = (\mathfrak{v}_X^2)^n = i_*(w^n) = \sum_{k=0}^{n} \binom{n}{k} i_*(a^k \times b^{n-k}).$$

If n is odd each term in the last sum vanishes for dimensional reasons. Suppose now that n is even, $n = 2m$. Then we have

$$\mathfrak{v}_X^{2n} = \binom{2m}{m} i_*(a^m \times b^m).$$

We mentioned in the proof of Theorem 3.1.1 that the binomial coefficient $\binom{2m}{m}$ is always even. Since $6i_*(a^m \times b^m) = 0$ we just need to assure that $\binom{2m}{m}$ is also divisible by 3; which is the case if the 3-adic expansion of m contains at least one digit 2, see [21], Lemma 19. Therefore $\mathfrak{v}_X^{2n} = 0$ under the assumptions indicated in the first statement of Theorem 3.1.3. By Theorem 3.2.3 this proves the first part of the theorem.

Next we prove the second statement of Theorem 3.1.3. Let $n \geq 1$ be such that its 3-adic expansion contains only digits 0 and 1. Then the binomial coefficient $\binom{2n}{n}$ is not divisible by 3, see [21], Lemma 19.

Consider the lens space $L_3^{2n+1} = S^{2n+1}/\mathbb{Z}_3$ where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the unit sphere and $\mathbb{Z}_3 = \{1, \xi, \xi^2\}$ acts as the group of roots of 1, where $\xi = \exp 2\pi i/3$. It is well known that the lens space has a cell decomposition with a unique cell in every dimension i for $i = 0, 1, \dots, 2n+1$, see [34], page 144-145. Let X the $2n$-skeleton of L_3^{2n+1}. Note that X has homotopy type of the lens space L_3^{2n+1} with one point removed. We will prove that $\mathsf{TC}(X) = 4n+1$ using the technique developed in [21].

The cohomology algebra $H^*(L_3^{2n+1}; \mathbb{Z}_3)$ can be described as the quotient of the polynomial algebra $\mathbb{Z}_3[x, y]$ with two generators x of degree 1 and y of degree 2 subject to relations $x^2 = 0$, $y^{n+1} = 0$ and $xy^n = 0$; consult [34], page 251. Here x is the generator of $H^1(X; \mathbb{Z}_3)$ and $y = \beta(x) \in H^2(X; \mathbb{Z}_3)$ is the image of x under the Bockstein homomorphism

$$\beta : H^1(X; \mathbb{Z}_3) \to H^2(X; \mathbb{Z}_3)$$

corresponding to the exact sequence

$$0 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 0.$$

The classes y^k and xy^k, where $k = 0, 1, \dots, n$, form an additive basis of $H^*(X; \mathbb{Z}_3)$. By the Künneth theorem one has

$$H^*(X \times X; \mathbb{Z}_3) = H^*(X; \mathbb{Z}_3) \otimes H^*(X; \mathbb{Z}_3)$$

and therefore the classes

$$x^a y^b \times x^c y^d \in H^*(X \times X; \mathbb{Z}_3)$$

where $a, c \in \{0, 1\}$ and $b, d \in \{0, 1, \dots, n\}$ and $(a, b) \neq (1, n)$, $(c, d) \neq (1, n)$ form an additive basis of $H^*(X \times X; \mathbb{Z}_3)$. We represent by $\bar{x}$ and $\bar{y}$ the classes

$$\bar{x} = x \times 1 - 1 \times x \in H^1(X \times X; \mathbb{Z}_3), \quad \bar{y} = y \times 1 - 1 \times y \in H^2(X \times X; \mathbb{Z}_3).$$

It is shown in [21] that $\beta(\bar{x}) = \bar{y}$ and therefore the class $\bar{y}$ has weight two with respect to the path fibration (2.1).

Recall Definition 2.4.1: a cohomology class $u \in H^*(X \times X; R)$ has weight greater or equal than k if $u|_A = 0$ for any open subset $A \subset X \times X$ with $\mathsf{TC}_X(A) \leq k$; see.

By Lemma 2.4.5 one has

$$\mathrm{wgt}((\bar{y})^{2n}) \geq 2n \cdot \mathrm{wgt}(\bar{y}) \geq 4n.$$

Adding Proposition 2.4.3 implies that if $(\bar{y})^{2n} \neq 0$ then $\mathsf{TC}(X) \geq 4n + 1$. A simple computation shows that

$$(\bar{y})^{2n} = (-1)^n \binom{2n}{n} y^n \times y^n$$

and the binomial coefficient $\binom{2n}{n}$ is mutually prime to 3 due to the fact that the 3-adic expansion of n does not contain any 2; we refer once more to Lemma 19 from [21]. We obtain $(\bar{y})^{2n} \neq 0$ which shows that $\mathsf{TC}(X) \geq 4n + 1$.

The opposite inequality $\mathsf{TC}(X) \leq 4n+1$ follows directly from the general upper bound $\mathsf{TC}(X) \leq 2\dim X + 1$. Hence

$$\mathsf{TC}(X) = 4n + 1.$$

Chapter 4

Navigation Functions

In [36] and [37] the authors explored the idea of using the gradient flow of Morse functions to develop motion planners which allow navigation of a mechanism to a fixed goal. In [20], Farber introduced a similar technique which produces motions connecting arbitrary points on a manifold M. In this variation the correct idea is to study the gradient flow of certain Morse-Bott functions on $M \times M$. Such functions are called *navigation functions*. These can be used to construct motion planning algorithms. Through this method one obtains upper bounds of $\mathsf{TC}(M)$; see Theorem 4.1.1 below. In this chapter we introduce and study a class of navigation functions on projective and lens spaces.

4.1 Navigation functions as motion planners

We start by reviewing some definitions regarding smooth maps. Let M be a closed manifold. A *Morse function* is a smooth map $f : M \to \mathbb{R}$ such that every critical point p is nondegenerate, *i.e.* the Hessian matrix of f at p is nonsingular. If the set of critical points is a disjoint union of connected submanifolds of M (called *critical* *submanifolds*) and the Hessian of f is nondegenerate on the normal bundle to each of the critical submanifolds (the quotient of the tangent bundle of M with the tangent bundle of the critical submanifold), then we say that f is a *Morse-Bott function.* If p is a critical point the *index of* p, $I(p)$, is the number of negative eigenvalues of

$$Hess_p f = \left(\left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_p \right)_{i,j},$$

the Hessian matrix of f at p. If N is a connected critical submanifold then every $p \in N$ has identical index and we define the index of N as $I(N) := I(p)$.

The negative gradient flow of f,

$$\Phi : \mathbb{R} \times M \to M,$$

is the flow associated to the tangent vector field $-\nabla f$. For a critical submanifold N define the sets

$$\mathcal{U}(N) = \{x \in M | \lim_{t=-\infty} \Phi(t,x) = q, q \in N\}$$

and

$$\mathcal{S}(N) = \{x \in M | \lim_{t=+\infty} \Phi(t,x) = q, q \in N\}.$$

It is well known that if f is a Morse-Bott function then $\mathcal{U}(N)$ and $\mathcal{S}(N)$ are manifolds; the unstable and stable manifolds, respectively. In fact, $\mathcal{U}(N)$ and $\mathcal{S}(N)$ are fibre bundles over N with fibres diffeomorphic to $\mathbb{R}^{I(N)}$ and $\mathbb{R}^{m-n-I(N)}$, where $m = \dim M$ and $n = \dim N$. Let $N_1, \ldots, N_k$ denote all the critical submanifolds of f. Then

$$M = \bigcup_{i=1}^{k} \mathcal{U}(N_i) = \bigcup_{i=1}^{k} \mathcal{S}(N_i) \tag{4.1}$$

Moreover, $\mathcal{U}(N_i) \cap \mathcal{U}(N_j) = \emptyset = \mathcal{S}(N_i) \cap \mathcal{S}(N_j)$ for $i \neq j$.

Let $i : N \to M$ be the natural inclusion of N into M. The normal bundle to N in M is the quotient

$$\nu(N) = \frac{i^*TM}{TN},$$

where i^*TM is the restriction of the tangent bundle TM by the inclusion i. If f is Morse-Bott then the flow lines are 'transversal' to the critical submanifolds. In fact for any critical submanifold N, the total space of the fibre bundle $\mathcal{U}(N) \oplus \mathcal{S}(N)$ is homeomorphic to the total space of the normal bundle $\nu(N)$ in M.

We now introduce the key definition of this chapter.

Definition 4.1.1. *Let M be a manifold. A smooth map $f : M \times M \to \mathbb{R}$ is called a* ***navigation function*** *if the following assumptions are verified:*

1. *$f(x, y) \geq 0$ for all $x, y \in M$,*

2. *$f(x, y) = 0$ if and only if $x = y$,*

3. *f is a Morse-Bott function.*

Remark 4.1.1. Observe that condition 2) implies that the diagonal of $M \times M$, *i.e.*

$$\Delta M = \{(x, y) \in M \times M : x = y\},$$

is always a critical submanifold of any navigation function on M.

The connection between navigation functions and topological complexity is made explicit by the next result, Theorem 4.32 from [20]. For convenience of the reader we reproduce the proof.

Theorem 4.1.1 ([20]). *Let $f : M \times M \to \mathbb{R}$ be a navigation function for M with critical submanifolds $N_1, \ldots, N_k \subset M \times M$. Denote by c_i the respective critical* *values,* i.e., $f(N_i) = \{c_i\}$. *Then*

$$\mathsf{TC}(M) \leq \sum_{r \in \text{ Crit}(f)} \mathcal{N}_r,$$

where

$$\mathcal{N}_r = \max_{c_i = r} \{\mathsf{TC}_M(N_i)\}$$

and $\text{Crit}(f) \subset \mathbb{R}$ *denotes the set of critical values.*

Proof. Let Φ denote the flow associated with the tangent vector field $-\nabla f$. By (4.1) we have a decomposition by stable manifolds of critical submanifolds, *i.e.*,

$$M \times M = \bigcup_{i=1}^{k} \mathcal{S}(N_i).$$

For every i, Φ induces a continuous retraction

$$\mathsf{q}_i : \mathcal{S}(N_i) \to N_i$$

from the stable manifold of N_i onto N_i given by

$$\mathsf{q}_i(x, y) = \lim_{t \to +\infty} \Phi(t, (x, y)), \quad (x, y) \in M \times M.$$

Let $f(S_i) = \{c_i\}$. Given a critical value r, denote by C_r the union

$$C_r = \bigcup_{c_i = r} N_i.$$

Clearly

$$\mathcal{S}(C_r) = \mathcal{S}\left(\bigcup_{c_i = r} N_i\right) = \bigcup_{c_i = r} \mathcal{S}(N_i).$$

It is well known that if $f(N_i) = f(N_j)$ for $i \neq j$ then $\overline{\mathcal{S}(N_i)} \cap \mathcal{S}(N_j) = \emptyset$. Hence there is a continuous retraction $\mathcal{Q}_r : \mathcal{S}(C_r) \to C_r$ such that $\mathcal{Q}_r|_{N_i} = q_i$. Moreover, Proposition 2.3.1 implies that

$$\mathsf{TC}_M(\mathcal{S}(C_r)) = \mathsf{TC}_M\left(\bigcup_{c_i = r} N_i\right) = \max_{c_i = r} \mathsf{TC}_M(N_i) = \mathcal{N}_r.$$

Applying the statement of Remark 2.3.1 with respect to the partition

$$M \times M = \bigcup_{r \in \ \mathrm{Crit}(f)} \mathcal{S}(C_r).$$

proves the inequality

$$\mathsf{TC}(M) \leq \sum_{r \in \ \mathrm{Crit}(f)} \mathsf{TC}_M(\mathcal{S}(C_r)) = \sum_{r \in \ \mathrm{Crit}(f)} \mathcal{N}_r.$$

$\square$

We see that the problem of global motion planning on M reduces to constructing sections of the path fibration $PM \to M \times M$ over the critical submanifolds.

4.2 A navigation function on lens spaces

We recall the construction of lens spaces. Let $\xi = e^{\frac{2\pi}{m} i} \in \mathbb{C}$. The multiplication by ξ defines a $\mathbb{Z}_m$-action on $\mathbb{C}^n$. The $(2n-1)$-dimensional sphere, S^{2n-1}, is naturally embedded in $\mathbb{C}^n$; it is $\mathbb{Z}_m$-invariant and $\mathbb{Z}_m$ acts freely on S^{2n-1}. The quotient of S^{2n-1} by this action is the lens space

$$L_m^{2n-1} = S^{2n-1}/\mathbb{Z}_m.$$

Given a point $z \in S^{2n-1}$, we denote by $[z]$ the image of z under the projection $S^{2n-1} \to L_m^{2n-1}$.

The main goal of this section is to study the number $\mathsf{TC}(L_m^{2n-1})$. We introduce a navigation function for the lens space L_m^{2n-1} and apply Theorem 4.1.1 with respect to this function.

Consider first the function $\widetilde{F} : S^{2n-1} \times S^{2n-1} \to \mathbb{R}$ defined by

$$\widetilde{F}(z, z') = \prod_{j=0}^{m-1} |z - \xi^j z'|^2, \quad z, z' \in S^{2n-1}. \tag{4.2}$$

It is clear that this map is smooth, symmetric, and invariant under the $\mathbb{Z}_m \times \mathbb{Z}_m$-action on $S^{2n-1} \times S^{2n-1}$ given by

$$(\xi^j, \xi^k) \cdot (u, v) = (\xi^j u, \xi^k v).$$

The product space $L_m^{2n-1} \times L_m^{2n-1}$ is the $\mathbb{Z}_m \times \mathbb{Z}_m$-quotient of $S^{2n-1} \times S^{2n-1}$. Thus, $\widetilde{F}$ induces a smooth function

$$F : L_m^{2n-1} \times L_m^{2n-1} \to \mathbb{R} \tag{4.3}$$

given by $F([z],[z']) = \widetilde{F}(z, z')$. Clearly F satisfies properties 1) and 2) of Definition 4.1.1. Property 3) will be verified later in Proposition 4.2.8. Thus F is a navigation function. The critical points of F will be described by Proposition 4.2.1. We first introduce a space that plays a key role in that result.

Consider the complex Stiefel manifold $V_2(\mathbb{C}^n)$, *i.e.* the space of pairs of orthonormal vectors in $\mathbb{C}^n$ with respect to the Hermitian inner product

$$\langle z, z' \rangle = \sum_{i=1}^{n} z_i \bar{z}'_i \in \mathbb{C}.$$

By definition $V_2(\mathbb{C}^n)$ is a submanifold of $S^{2n-1} \times S^{2n-1}$. Due to the properties of the Hermitian inner product it follows that, for any $z, z' \in S^{2n-1} \subset \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C} \backslash \{0\}$, one has $\langle z, z' \rangle = 0$ if and only if $\langle \alpha z, \beta z' \rangle = 0$. In particular, we have a well defined quotient manifold

$$V_L = V_2(\mathbb{C}^n)/\mathbb{Z}_m \times \mathbb{Z}_m.$$

Proposition 4.2.1. *Let $m \geq 3$ and $L = L_m^{2n-1}$ be the lens space described above. The critical submanifolds of the navigation function $F : L \times L \to \mathbb{R}$ induced by (4.2) are the following:*

a) $\Delta_L = \{([z],[z']) \in L \times L : [z] = [z']\}$, *the diagonal;*

b) $\Delta'_L = \{([z],[z']) \in L \times L : [z] = [e^{\frac{\pi}{m}i} z']\}$, *the "shifted" diagonal;*

c) $V_L = V_2(\mathbb{C}^n)/\mathbb{Z}_m \times \mathbb{Z}_m = \{([z],[z']) \in L \times L : \langle z, z' \rangle = 0\}$, *the "orthonormal" 2-frames on L.*

Remark 4.2.1. Simple calculations show that the critical values are the following:

$$F(\Delta_L) = 0; \quad F(\Delta'_L) = 2^m \cdot \prod_{j=1}^{m} \sin\left(\frac{(2j-1)\pi}{2m}\right); \quad F(V_L) = 2^m.$$

Hence

$$F(\Delta_L) < F(\Delta'_L) < F(V_L).$$

4.2.1 Proof of Proposition 4.2.1

Let $T_z S^{2n-1}$ be the tangent space at z of the sphere S^{2n-1}. Identify $T_z S^{2n-1}$ with the set $\{v \in \mathbb{C}^n | \ \mathsf{Re}\langle v, z\rangle = 0\}$. Given a point $(z, z') \in S^{2n-1} \times S^{2n-1}$ we assume the identification $T_{(z,z')}(S^{2n-1} \times S^{2n-1}) \simeq T_z S^{2n-1} \times T_{z'} S^{2n-1}$.

For any $v \in T_z S^{2n-1}$ one has

$$\begin{aligned}
\left.\frac{\partial \widetilde{F}}{\partial(v,0)}\right|_{(z,z')} &= \sum_{j=0}^{m-1} \left(\left(\frac{\partial}{\partial(v,0)} |z - \xi^j z'|^2 \right) \cdot \prod_{k \neq j} |z - \xi^k z'|^2 \right) \\
&= \sum_{j=0}^{m-1} \left(\left(\frac{\partial}{\partial(v,0)} \left(-2\mathsf{Re}\left(\langle z, \xi^j z'\rangle\right)\right) \right) \cdot \prod_{k \neq j} |z - \xi^k z'|^2 \right) \\
&= -2 \sum_{j=0}^{m-1} \left(\mathsf{Re}\left(\langle v, \xi^j z'\rangle\right) \cdot \prod_{k \neq j} |z - \xi^k z'|^2 \right).
\end{aligned}$$

Let

$$\widetilde{\Delta} = \{(z, z') \in S^{2n-1} \times S^{2n-1} | \ z' = \xi^j z, \ j = 0, \dots, m-1\}. \tag{4.4}$$

If $(z, z') \in \widetilde{\Delta}$ then $z' = \xi^j z$, for some $j \in \{0, 1, \dots, m-1\}$. Thus

$$\left.\frac{\partial \widetilde{F}}{\partial(v,0)}\right|_{\widetilde{\Delta}} = -2\mathsf{Re}\left(\langle v, z\rangle\right) \cdot \left.\prod_{k \neq j} |z - \xi^k z'|^2 \right|_{\widetilde{\Delta}} = 0.$$

Analogously,

$$\left.\frac{\partial \widetilde{F}}{\partial(0,w)}\right|_{\widetilde{\Delta}} = 0$$

for any $w \in T_{z'}S^{2n-1}$. On the other hand, if $z \neq \xi^j z'$ for any $j = 0, 1, \ldots, m-1$, we have

$$\frac{\partial \widetilde{F}}{\partial v} = -2\widetilde{F} \cdot \mathsf{Re}\left(\langle v, \mu z' \rangle\right), \tag{4.5}$$

where $\mu = \mu(z, z')$ is defined by

$$\mu(z, z') = \sum_{j=0}^{m-1} \frac{\xi^j}{|z - \xi^j z'|^2}. \tag{4.6}$$

Similar computations show that for $w \in T_{z'}S^{2n-1}$ one has

$$\frac{\partial \widetilde{F}}{\partial w} = -2\widetilde{F} \cdot \mathsf{Re}\left(\langle z, \mu w \rangle\right) = -2\widetilde{F} \cdot \mathsf{Re}\left(\langle w, \bar{\mu} z \rangle\right). \tag{4.7}$$

One can naturally identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Given $x, y \in \mathbb{C}^n$, denote by x^*, y^* the associated real vectors. It is easy to verify that $\mathsf{Re}\left(\langle x, y \rangle\right) = \langle x^*, y^* \rangle_{\mathbb{R}}$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denotes the usual real inner product. Denote by $P_T(\cdot) : \mathbb{R}^l \to T$ the real orthogonal projection onto a real vector subspace $T \subset \mathbb{R}^l$. By (4.5) and (4.7) we can express the gradient $\nabla \widetilde{F}$ at a point $(z, z') \in S^{2n-1} \times S^{2n-1} - \widetilde{\Delta}$ as

$$\nabla \widetilde{F}(z, z') = -2\widetilde{F} \cdot P_{T \times T'}((\mu z')^*, (\bar{\mu} z)^*), \tag{4.8}$$

where $T = T_z S^{2n-1} \subset \mathbb{R}^{2n}$ and $T' = T_{z'} S^{2n-1} \subset \mathbb{R}^{2n}$.

One can now use the formula (4.8) to describe the critical submanifolds of $\widetilde{F}$.

Corollary 4.2.2. *A point $(z, z') \in S^{2n-1} \times S^{2n-1}$ is a critical point of $\widetilde{F}$ if and only if one of the following holds:*

1. $\widetilde{F}(z, z') = 0$;

2. *$\widetilde{F}(z,z') \neq 0$ and $\mu(z,z') = 0$;*

3. *$\widetilde{F}(z,z') \neq 0$, $\mu(z,z') \neq 0$ and $\mu(z,z')z' = \lambda z$ for some $\lambda \in \mathbb{R}$.*

Proof. We already saw that all points in $\widetilde{\Delta}$, given by (4.4), are critical points of $\widetilde{F}$. Clearly $\widetilde{F}(z,z') = 0$ if and only if $(z,z') \in \widetilde{\Delta}$. Thus all points in the solution set of $\widetilde{F}(z,z') = 0$ are critical. By (4.8), any critical point (z,z') such that $\widetilde{F}(z,z') \neq 0$ must satisfy

$$P_{T\times T'}((\mu z')^*, (\bar{\mu} z)^*) = 0, \tag{4.9}$$

where μ is as defined in (4.6). Clearly, points (z,z') such that $\mu(z,z') = 0$ are solutions of this equation. If (z,z') is a solution of equation (4.9) and $\mu(z,z') \neq 0$ then $P_T((\mu z')^*) = 0$, *i.e.*

$$\mu z' = \lambda_1 z, \tag{4.10}$$

for some $\lambda_1 \in \mathbb{R}$, and $P_{T'}((\bar{\mu} z)^*) = 0$, which is equivalent to

$$\bar{\mu} z = \lambda_2 z', \tag{4.11}$$

for some $\lambda_2 \in \mathbb{R}$. However, multiplying both sides of (4.11) by $\frac{\mu}{\lambda_2}$ gives $\lambda_3 z = \mu z'$, with $\lambda_3 = \frac{|\mu|^2}{\lambda_2}$. Therefore conditions (4.10) and (4.11) are equivalent.

We conclude that a critical point (z,z') of the function $\widetilde{F}$ must satisfy either

$$\widetilde{F}(z,z') = 0$$

or

$$\mu(z,z') = 0$$

or

$$\mu(z,z')z' = \lambda z,$$

for some $\lambda \in \mathbb{R}$. $\square$

The next goal is to use Corollary 4.2.2 to describe explicitly the critical submanifolds of $\widetilde{F}$.

Proposition 4.2.3. *For $z, z' \in S^{2n-1}$ such that $\widetilde{F}(z,z') \neq 0$ one has $\mu(z,z') = 0$ if and only if $\langle z, z' \rangle = 0$.*

Proposition 4.2.4. *For $z, z' \in S^{2n-1}$ such that $\widetilde{F}(z,z') \neq 0$ and $\mu(z,z') \neq 0$ one has $\mu(z,z')z' = \lambda z$ for some $\lambda \in \mathbb{R}$ if and only if $z = e^{i\theta} z'$, where $\theta = \frac{(2k+1)\pi}{m}$ for some integer k.*

For the proofs of the above statements we need to introduce some definitions. Given $w \in \mathbb{C}$, define the sets

$$H_+(w) = \{\alpha w | \ \alpha \in \mathbb{C}, \ \mathsf{Im}(\alpha) > 0\},$$

$$H_-(w) = \{\alpha w | \ \alpha \in \mathbb{C}, \ \mathsf{Im}(\alpha) < 0\},$$

$$J_+(w) = \{j \in \mathbb{Z} | \ 0 \leq j \leq m-1, \ \xi^j \in H_+(w)\},$$

$$J_-(w) = \{j \in \mathbb{Z} | \ 0 \leq j \leq m-1, \ \xi^j \in H_-(w)\},$$

$$J_0(w) = \{j \in \mathbb{Z} | \ 0 \leq j \leq m-1, \ \xi^j \in \mathbb{R}w\}.$$

The sets $H_+(w)$ and $H_-(w)$ are the two half-planes in $\mathbb{C}$ separated by the real line

$$\mathbb{R}w = \{\lambda w | \ \lambda \in \mathbb{R}\};$$

see Figure 4.1.

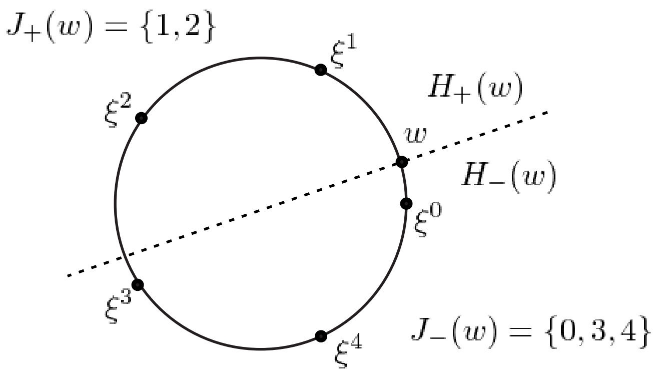

Figure 4.1: The sets $J_-(w)$ and $J_+(w)$.

Proof of Propostion 4.2.3. There is a bijection between $J_+(1)$ and $J_-(1)$ given by $j \mapsto m - j$. Hence, we may rewrite (4.6) as

$$\mu(z, z') = \sum_{j \in J_+(1)} \left(\frac{\xi^j}{|z - \xi^j z'|^2} + \frac{\xi^{-j}}{|z - \xi^{-j} z'|^2} \right) + \sum_{j \in J_0(1)} \frac{\xi^j}{|z - \xi^j z'|^2}. \tag{4.12}$$

By definition of $J_0(1)$, the second sum on the right side of (4.12) is a real number. Thus, in order to have $\mu(z, z') = 0$, the sum in (4.12) whose indices run over $J_+(1)$ must also be a real number.

Lemma 4.2.5. *Suppose $z, z' \in S^{2n-1} \subset \mathbb{C}^n$ are such that $\langle z, z' \rangle \in H_\pm(1)$. Then, for any $j \in J_+(1)$,*

$$\frac{\xi^j}{|z - \xi^j z'|^2} + \frac{\xi^{-j}}{|z - \xi^{-j} z'|^2} \in H_\pm(1). \tag{4.13}$$

In particular, by formula (4.12), $\mu(z, z')$ is nonzero.

Proof. Assume that $\langle z, z'\rangle \in H_+(1)$ and $j \in J_+(1)$, *i.e.* $\mathsf{Im}(\langle z, z'\rangle), \mathsf{Im}(\xi^j) > 0$. Then

$$\begin{aligned} |z - \xi^j z'|^2 - |z - \xi^{-j} z'|^2 &= -\xi^{-j}\langle z, z'\rangle - \xi^j\langle z', z\rangle + \xi^j\langle z, z'\rangle + \xi^{-j}\langle z', z\rangle \\ &= \left(\xi^j - \xi^{-j}\right)\left(\langle z, z'\rangle - \langle z', z\rangle\right) \\ &= \left(\xi^j - \bar{\xi^j}\right)\left(\langle z, z'\rangle - \overline{\langle z, z'\rangle}\right) \\ &= -4\mathsf{Im}(\xi^j)\mathsf{Im}(\langle z, z'\rangle) \\ &< 0, \end{aligned}$$

since, for any $\alpha \in \mathbb{C}$, $\alpha - \bar{\alpha} = 2\mathsf{Im}(\alpha)\cdot i$. Obviously $0 < \mathsf{Im}(\xi^j) = -\mathsf{Im}(\xi^{-j})$. Since $|z - \xi^j z'|^2 < |z - \xi^{-j} z'|^2$ then

$$\mathsf{Im}\left(\frac{\xi^j}{|z - \xi^j z'|^2}\right) + \mathsf{Im}\left(\frac{\xi^{-j}}{|z - \xi^{-j} z'|^2}\right) > 0,$$

i.e. the sum (4.13) must lie in $H_+(1)$. The proof is analogous in the case $\langle z, z'\rangle \in H_-(1)$. □

We have just seen that if $\mu(z, z') = 0$ then $\langle z, z'\rangle \in \mathbb{R}$. On the other hand,

$$\mu(\xi z, z') = \sum_{j=0}^{m-1} \frac{\xi^j}{|\xi z - \xi^j z'|^2} = \sum_{j=0}^{m-1} \frac{\xi^j}{|z - \xi^{j-1} z'|^2} = \xi\mu(z, z').$$

Hence,

$$\mu(z, z') = 0 \Leftrightarrow \mu(\xi z, z') = 0 \Rightarrow \langle \xi z, z'\rangle \in \mathbb{R} \Leftrightarrow \langle z, z'\rangle \in \mathbb{R}\bar{\xi}.$$

This shows that if $\mu(z, z') = 0$ then $\langle z, z'\rangle \in \mathbb{R} \cap \mathbb{R}\bar{\xi} = \{0\}$.

The inverse statement is clearly true. Indeed, the condition $\langle z, z'\rangle = 0$ implies that all the denominators of the summands on the right side of equation (4.6) are equal. Since $\sum_{j=0}^{m-1} \xi^j = 0$ it follows that $\mu(z, z') = 0$.

This completes the proof of Proposition 4.2.3. □

The proof of Proposition 4.2.4 follows the same idea of the above proof but it is technically more delicate.

Proof of Propostion 4.2.4. Any two points $z, z' \in S^{2n-1}$ which satisfy the hypothesis of Proposition 4.2.4 must lie in the same complex line. Let $w \in S^1 \subset \mathbb{C}$ be the complex number defined by

$$z = wz'.$$

Since $\widetilde{F}(z, z') \neq 0$, the complex number w cannot lie in the subgroup generated by ξ which we denote by

$$O_m = \{e^{\frac{2k\pi}{m}i} |\ k = 0, 1, \dots, m-1\} < S^1. \tag{4.14}$$

The existence of a nonzero real number λ such that $\mu(z, z')z' = \lambda z$ is equivalent to

$$\sum_{j=0}^{m-1} \frac{\xi^j}{|w - \xi^j|^2} \in \mathbb{R}w. \tag{4.15}$$

Denote by O'_m the coset $e^{\frac{\pi}{m}i} \cdot O_m$ of the subgroup $O_m < S^1 \subset \mathbb{C}$. Explicitly,

$$O'_m = \{e^{\frac{(2k+1)\pi}{m}i}, k = 0, 1, \dots, m-1\}. \tag{4.16}$$

We will show that (4.15) holds if and only if $w \in O'_m$. This will be a consequence of Lemma 4.2.6. First we need some preparation.

Let us start by rewriting the sum in (4.15) as

$$\sum_{j=0}^{m-1} \frac{\xi^j}{|w - \xi^j|^2} = \sum_{j \in J_+(w) \cup J_-(w)} \frac{\xi^j}{|w - \xi^j|^2} + \sum_{j \in J_0(w)} \frac{\xi^j}{|w - \xi^j|^2}. \tag{4.17}$$

Since the last sum in (4.17) takes value in $\mathbb{R}w$, statement (4.15) holds if and only if

$$\sum_{j \in J_+(w) \cup J_-(w)} \frac{\xi^j}{|w - \xi^j|^2} \in \mathbb{R}w. \tag{4.18}$$

Assume, without loss of generality that

$$\min_{j \in J_-(w)} |w - \xi^j| \leq \min_{j \in J_+(w)} |w - \xi^j|. \tag{4.19}$$

If necessary, this assumption is true by the interchange of coordinates $(z, z') \to (z', z)$ since $\widetilde{F}$ is symmetric. Let us reorder the indices of the sets $J_+(w)$ and $J_-(w)$ by increasing norm of $w - \xi^j$; for example, if w is in the short arc delimited by ξ^j and ξ^{j+1}, then j is the first element of $J_-(w)$ and $j+1$ is the first element of $J_+(w)$. Notice that, under the assumption of (4.19), the cardinalities of sets $J_+(w)$ and $J_-(w)$ is either the same or $|J_-(w)| = |J_+(w)| + 1$; Figure 4.2 illustrates both cases.

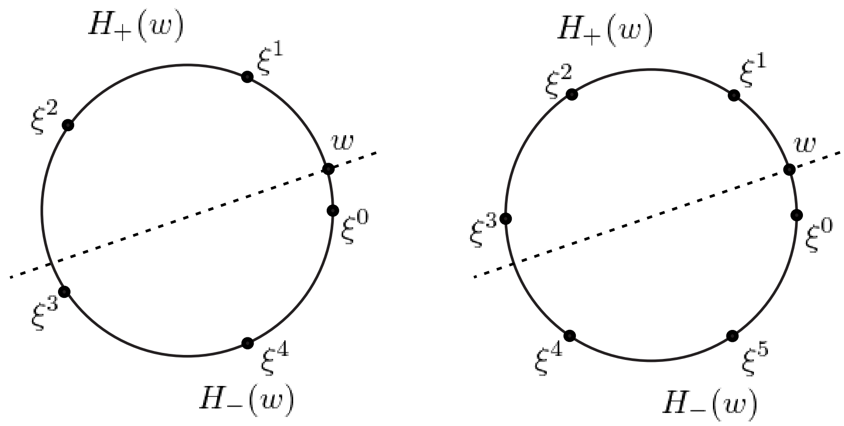

Figure 4.2: Possible configuration for $m = 5$ (left) and $m = 6$ (right). The elements of $J_-(w)$ and $J_+(w)$ are the respective exponents in $H_-(w)$ and $H_+(w)$.

Denote by

$$\phi : J_+(w) \to J_-(w)$$

the injective map which associates the *k-th* element of $J_+(w)$ with the *k-th* element of $J_-(w)$. As example notice that in the left side case of Figure 4.2 one has

$$J_-(w) = \{0, 3, 4\}, \quad J_+(w) = \{1, 2\}$$

and

$$\phi(1) = 0, \quad \phi(2) = 4.$$

On the right side configuration we have $J_-(w) = \{0, 4, 5\}$ and $J_+(w) = \{1, 2, 3\}$ and

$$\phi(1) = 0, \quad \phi(2) = 5, \quad \phi(3) = 4.$$

Remark 4.2.2. *The map ϕ was already introduced in the proof of Proposition 4.2.3. There it was described as the bijection between $J_+(1)$ and $J_-(1)$ given by $j \to m - j$.*

Define R to be the following complex number. If m is even or m is odd with $w \in O'_m$ set

$$R = 0.$$

If m is odd and $w \notin O'_m$ then ϕ is not a bijection; by (4.19), it follows that $|J_-(w)| = |J_+(w)| + 1$. In this case set

$$R = \frac{\xi^l}{|w - \xi^l|^2} \in H_-(w), \tag{4.20}$$

where l is the last element of $J_-(w)$, with respect to the ordering defined above. In the situation pictured on the left side of Figure 4.2 we have $R = \frac{\xi^3}{|w-\xi^3|^2}$. For the configuration on the right $R = 0$.

The next Lemma plays the same role that Lemma 4.2.5 played in the proof of Proposition 4.2.3.

Lemma 4.2.6. *Let O_m and O'_m be the sets defined in (4.14) and (4.16), respectively, and $w \in S^1 \backslash O_m \subset \mathbb{C}$.*

1. *If $w \in O'_m$ then ϕ is the bijection given by reflection on $\{\lambda w | \ \lambda \in \mathbb{R}\}$. In particular, for any $j \in J_+(w)$*

$$\frac{\xi^j}{|w - \xi^j|^2} + \frac{\xi^{\phi(j)}}{|w - \xi^{\phi(j)}|^2} \in \mathbb{R}w.$$

2. If $w \notin O'_m$, i.e. $w \neq e^{\frac{k\pi}{m}i}$ for $k = 0, \ldots, m-1$, and

$$\min_{j \in J_-(w)} |w - \xi^j| < \min_{j \in J_+(w)} |w - \xi^j|, \tag{4.21}$$

then, for any $j \in J_+(w)$,

$$\frac{\xi^j}{|w - \xi^j|^2} + \frac{\xi^{\phi(j)}}{|w - \xi^{\phi(j)}|^2} \in H_-(w). \tag{4.22}$$

The first part of the lemma implies that if $w \in O'_m$ then (4.18) and consequently (4.15) hold. Hence, for any pair $(z, z') \in \widetilde{\Delta}$, where

$$\widetilde{\Delta'} = \{(z, z') \in S \times S \mid z = e^{(2k+1)\frac{\pi}{m}i} z',\ k = 0, \ldots, m-1\},$$

there is a nonzero $\lambda \in \mathbb{R}$ such that

$$\mu(z, z')z' = \lambda z.$$

The restriction (4.21) in the second part of the lemma, is equivalent to (4.19) since equality can only occur when $w \in O'_m$.

Since

$$\sum_{j \in J_+(w) \cup J_-(w)} \frac{\xi^j}{|w - \xi^j|^2} = \sum_{j \in J_+(w)} \left(\frac{\xi^j}{|w - \xi^j|^2} + \frac{\xi^{\phi(j)}}{|w - \xi^{\phi(j)}|^2} \right) + R,$$

we conclude that if $w \notin O'_m$ then (4.18) cannot hold since, by (4.22), the sum in (4.18) must take value in $H_-(w)$ and either $R = 0$ or $R \in H_-(w)$.

Proving Lemma 4.2.6 terminates the proof Proposition 4.2.4.

Proof of Lemma 4.2.6. For every $j \in J_+(w)$ define

$$X_j = |w - \xi^j|, \quad Y_j = |w - \xi^{\phi(j)}|$$

and $\alpha_j, \beta_j \in (0, \pi)$ denote the angles formed by w and ξ^j and w and $\xi^{\phi(j)}$, respectively; see Figure 4.3 above.

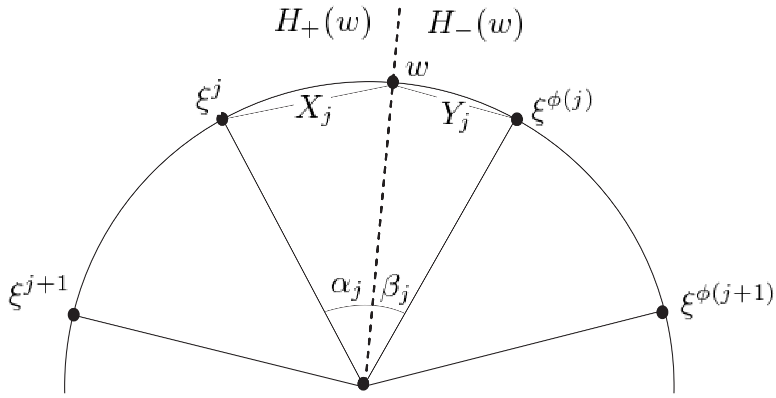

Figure 4.3: The numbers α_j, β_j, X_j and Y_j.

Clearly, the statement in the first part of the lemma holds. If $w \in O'_m$ then $w = e^{\frac{k\pi}{m}i}$ for some k. Thus, for any $j \in J_+(w)$, $X_j = Y_j$ and $\alpha_j = \beta_j$.

Let j_0 be the first element of $J_+(w)$ with respect to the ordering described before. Inequality (4.21) can be rewritten as $X_{j_0} > Y_{j_0}$. Clearly this implies that for any $j \in J_+(w)$ it holds that

$$X_j > Y_j \text{ and } \alpha_\mathrm{j} > \beta_\mathrm{j} \ .$$

Statement (4.22) is equivalent to

$$\frac{\sin \alpha_j}{X_j^2} < \frac{\sin \beta_j}{Y_j^2},$$

for any $j \in J_+(w)$. Since

$$X_j = 2\sin\left(\frac{\alpha_j}{2}\right) \quad , \quad Y_j = 2\sin\left(\frac{\beta_j}{2}\right)$$

the second part of the lemma follows from the following result.

Claim. *If* $\alpha, \beta \in (0, \pi)$ *such that* $\alpha > \beta$ *then*

$$Y^2 \sin \alpha < X^2 \sin \beta$$

where $X = \sin\left(\frac{\alpha}{2}\right)$, $Y = \sin\left(\frac{\beta}{2}\right)$.

The proof is straightforward. A property of the trignometric function *sin* states that

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

Thus

$$\begin{aligned} Y^2\sin(\alpha) < X^2\sin(\beta) \quad &\Leftrightarrow \quad \sin^2\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) < \sin^2\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) \\ &\Leftrightarrow \quad \tan\left(\frac{\beta}{2}\right) < \tan\left(\frac{\alpha}{2}\right) \\ &\Leftrightarrow \quad \beta < \alpha. \end{aligned}$$

□

This concludes the proof of Proposition 4.2.4.

□

Propositions 4.2.3 and 4.2.4 together with Corollary 4.2.2 imply the next result.

Corollary 4.2.7. *A critical point $(z, z') \in S^{2n-1} \times S^{2n-1}$ of the function $\widetilde{F}$ must satisfy one of the following conditions:*

1. $(z, z') \in \widetilde{\Delta}$, i.e. $z = e^{i\theta} z'$ *where* $\theta = \frac{2k\pi}{m}$ *for some integer* k*;*

2. $z = e^{i\theta} z'$ *where* $\theta = \frac{(2k+1)\pi}{m}$ *for some integer* k*;*

3. $\langle z, z' \rangle = 0$.

Corollary 4.2.7 provides a clear description of the critical submanifolds of the map $\widetilde{F}$. Each of the submanifolds is invariant by the $\mathbb{Z}_m \times \mathbb{Z}_m$-action and the respective quotients are the critical submanifolds of F. Namely, the quotients submanifolds defined in 1), 2) and 3) of the above Corollary are respectively the submanifolds Δ_L, Δ'_L and V_L introduced in Proposition 4.2.1. This terminates the proof of Propositon 4.2.1.

4.2.2 Proof of the Morse-Bott condition

Finally we prove that F is a Morse-Bott function and therefore a navigation function.

Proposition 4.2.8. *The map $F : L_m^{2n-1} \times L_m^{2n-1} \to \mathbb{R}$ given by (4.3) is a Morse-Bott function.*

Proof. Clearly F is a Morse-Bott function if and only if $\widetilde{F}$ is Morse-Bott function. Thus we must prove that the Hessian of $\widetilde{F}$ is nondegenerate on each of the normal bundles to the critical submanifolds in $S^{2n-1} \times S^{2n-1}$.

Denote by S the sphere S^{2n-1}. By Corollary 4.2.7, the submanifolds

$$\begin{aligned} \Delta &= \{(z, z') \in S \times S |\ z = z'\}, && (4.23) \\ \Delta' &= \{(z, z') \in S \times S |\ z = e^{\frac{\pi}{m}} \cdot z'\}, && (4.24) \\ \mathcal{V} &= \{(z, z') \in S \times S |\ \langle z, z' \rangle = 0\}. && (4.25) \end{aligned}$$

of $S \times S$ are critical. Furthermore, the $\mathbb{Z}_m \times \mathbb{Z}_m$-orbits of the points in Δ and Δ' contain all the critical points described in 1) and 2) of Corollary 4.2.7. Since $\widetilde{F}$ is $\mathbb{Z}_m \times \mathbb{Z}_m$-invariant, it suffices to show that the Hessian of $\widetilde{F}$ is nondegenerate on the normal bundles in $S^{2n-1} \times S^{2n-1}$ to Δ, Δ' and $\mathcal{V}$.

We start by computing the tangent bundles to Δ, Δ' and $\mathcal{V}$. The space $S \times S$ has a natural embedding in $\mathbb{C}^n \times \mathbb{C}^n$. The tangent space to $S \times S$ at a point $* = (z, z')$ is the real vector space

$$\{(v, v') \in \mathbb{C}^n \times \mathbb{C}^n |\ \mathsf{Re}\langle v, z \rangle = 0,\ \mathsf{Re}\langle v', z' \rangle = 0\}.$$

From (4.23), (4.24) and (4.25) we can see that the tangent spaces to Δ, Δ' and

$\mathcal{V}$, at a point $* = (z, z') \in S \times S$, are the real vector spaces:

$$\begin{aligned}
T_*\Delta &= \{(v, v') \in \mathbb{C}^n \times \mathbb{C}^n |\ v' = v, \mathsf{Re}(\langle v, z\rangle) = 0\}; \\
T_*\Delta' &= \{(v, v') \in \mathbb{C}^n \times \mathbb{C}^n |\ v' = e^{\frac{\pi}{m}i} \cdot v,\ \mathsf{Re}(\langle v, z\rangle) = 0\}; \\
T_*\mathcal{V} &= \{(v, v') \in \mathbb{C}^n \times \mathbb{C}^n |\ \langle v, z'\rangle + \langle z, v'\rangle = 0,\ \mathsf{Re}(\langle v, z\rangle) = 0 = \mathsf{Re}(\langle v', z'\rangle)\}.
\end{aligned}$$

Denote the respective normal spaces by $N_*\Delta$, $N_*\Delta'$ and N_*V. Clearly,

$$N_*\Delta = \{(v, v') \in \mathbb{C}^n \times \mathbb{C}^n |\ v' = -v, \mathsf{Re}(\langle v, z\rangle) = 0\}$$

and

$$N_*\Delta' = \{(v, v') \in \mathbb{C}^n \times \mathbb{C}^n |\ v' = -e^{\frac{\pi}{m}i} \cdot v,\ \mathsf{Re}(\langle v, z\rangle) = 0\}.$$

Given a point $* = (z, z') \in V$ we may choose a basis

$$\mathcal{B} = \left\{(v_1, v_1'), \ldots, (v_{2n-1}, v_{2n-1}'), (v_1, -v_1), \ldots, (v_{2n-1}, -v_{2n-1}')\right\},$$

of the real vector space $T_*(S \times S)$, where $\{v_1, \ldots, v_{2n-1}\}$ and $\{v_1', \ldots, v_{2n-1}'\}$ are orthonormal basis of T_zS and $T_{z'}S$, respectively, and such that

$$v_1 = z',\ v_2 = iz',\ v_1' = z \text{ and } v_2' = -iz.$$

By the condition $\langle v, z'\rangle + \langle z, v'\rangle = 0$, we see that the space $T_*\mathcal{V}$ is the real vector space generated by the base

$$\mathcal{B}_{\mathcal{T}} = \mathcal{B} \backslash \{(v_1, v_1'), (v_2, v_2')\}.$$

Hence

$$N_*\mathcal{V} = \{\alpha(v_1, v_1') + \beta(v_2, v_2') |\ \alpha, \beta \in \mathbb{R}\} \subset \mathbb{C}^n \times \mathbb{C}^n.$$

Lemma 4.2.9. *The Hessian of $\widetilde{F}$ is nondegenerate on the normal bundle $N\Delta$.*

Proof. Let $* = (z, z)$ be a point in Δ. Given a vector $V = (v, -v) \in N_*\Delta$ one has

$$\left.\frac{\partial \widetilde{F}}{\partial V}\right|_* = \sum_{j=0}^{m-1} \left.\frac{\partial A_j}{\partial V}\widetilde{F}_j\right|_*,$$

where

$$A_j = ||z - \xi^j z'||^2 \quad \text{and} \quad \widetilde{F}_j = \prod_{k \neq j} A_k.$$

Hence,

$$\left.\frac{\partial^2 \widetilde{F}}{\partial W \partial V}\right|_* = \sum_j \left.\left(\frac{\partial^2 A_j}{\partial W \partial V}\widetilde{F}_j + \frac{\partial A_j}{\partial V}\frac{\partial \widetilde{F}_j}{\partial W}\right)\right|_* \tag{4.26}$$

where $W = (w, -w) \in N_*\Delta$. Note that $A_0 = 0$ and $\widetilde{F}_j = 0$ for $j \neq 0$. Thus we can write (4.26) as

$$\left.\frac{\partial^2 \widetilde{F}}{\partial W \partial V}\right|_* = \left.\frac{\partial^2 A_0}{\partial W \partial V}\widetilde{F}_0\right|_* + \sum_j \left.\frac{\partial A_j}{\partial V}\frac{\partial \widetilde{F}_j}{\partial W}\right|_*. \tag{4.27}$$

We claim that the second sum on the right side of (4.27) is valued zero. In fact,

$$\frac{\partial \widetilde{F}_j}{\partial W} = \sum_{k \neq j} \frac{\partial A_k}{\partial W}\widetilde{F}_{j,k},$$

where $\widetilde{F}_{j,k} = \prod_{l \notin \{j,k\}} A_k$. One has $\widetilde{F}_{j,k} = 0$ if both $j, k \neq 0$. On the other hand, for any $V = (v, -v) \in N_*\Delta$, one has

$$\left.\frac{\partial A_0}{\partial V}\right|_* = -2\left.\frac{\partial}{\partial V}\mathsf{Re}(\langle z, z'\rangle)\right|_* = -2\ \mathsf{Re}(\langle v, z'\rangle - \langle z, v\rangle)|_* = 0,$$

since $\langle v, z'\rangle = \langle v, z\rangle = \overline{\langle z, v\rangle}$. Hence, for any j,

$$\left.\left(\frac{\partial A_j}{\partial V} \cdot \frac{\partial \widetilde{F}_j}{\partial W}\right)\right|_* = 0.$$

By (4.27) we have

$$\left.\frac{\partial^2 \widetilde{F}}{\partial W \partial V}\right|_* = \left.\frac{\partial^2 A_0}{\partial W \partial V}\widetilde{F}_0\right|_*.$$

On the other hand,

$$
\begin{aligned}
\left.\frac{\partial^2 A_0}{\partial W \partial V}\right|_* &= \left. -2\frac{\partial}{\partial W}\mathsf{Re}(\langle v, z'\rangle - \langle z, v\rangle)\right|_* \\
&= -2\ \mathsf{Re}(\langle v, -w\rangle - \langle w, v\rangle)|_* \\
&= 4\mathsf{Re}(\langle v, w\rangle).
\end{aligned}
$$

When analyzing the Hessian matrix of $\widetilde{F}$ at a critical point $* = (z, z)$, we may assume that $V = (v, -v)$ and $W = (w, -w)$ are such that the vectors v and w are chosen from an orthonormal basis of T_zS, *i.e.* they are both unitary and either $v = w$ or $\mathsf{Re}(\langle v, w\rangle) = 0$. For these coordinates the Hessian matrix of $\widetilde{F}$ at a point $* \in \Delta$ is a diagonal matrix with constant value 4 in the diagonal entries. □

Lemma 4.2.10. *The Hessian of $\widetilde{F}$ is nondegenerate on the normal bundle $N\Delta'$.*

Proof. Set $\alpha = e^{\frac{\pi}{m}i}$ and let $* = (z, \alpha z)$ be a point in Δ'. We adopt the notations A_j, $\widetilde{F}_j$ and $\widetilde{F}_{j,k}$ from the proof of the previous lemma. Recall that a vector $V \in \Delta'$ has the form $V = (v, -\alpha v)$.

As we have seen in the proof of Lemma 4.2.9 one has

$$
\left.\frac{\partial^2 \widetilde{F}}{\partial W \partial V}\right|_* = \sum_{j=0}^{m-1}\left.\left(\frac{\partial^2 A_j}{\partial W \partial V}\widetilde{F}_j + \frac{\partial A_j}{\partial V}\sum_{k\neq j}\frac{\partial A_k}{\partial W}\widetilde{F}_{j,k}\right)\right|_*, \tag{4.28}
$$

for any $V, W \in N_*\Delta'$.

Let us choose a basis

$$
\mathcal{B}' = \{(v_1, -\alpha v_1), \ldots, (v_{2n-1}, -\alpha v_{2n-1})\}
$$

for $N_*\Delta'$ such that $\{v_1, \ldots, v_{2n-1}\}$ is an orthonormal basis of T_zS and if $V = (v, -\alpha v)$ is an element of $\mathcal{B}'$, then either

$$
v = iz \quad \text{or} \quad \mathsf{Im}(v, z) = 0.
$$

One has

$$\begin{aligned}
\left.\frac{\partial A_j}{\partial V}\right|_* &= -2\frac{\partial}{\partial V}\mathsf{Re}(\langle z, \xi^j z'\rangle)\Big|_* \\
&= -2\mathsf{Re}(\langle v, \xi^j z'\rangle - \langle z, \alpha\xi^j v\rangle)\big|_* \\
&= -2\mathsf{Re}(\langle v, z\rangle \cdot (\overline{\alpha\xi^j} - \alpha\xi^j)) \\
&= 4\mathsf{Re}(\langle v, z\rangle \cdot \mathsf{Im}(\alpha\xi^j)\cdot i)) \\
&= -4\mathsf{Im}(\alpha\xi^j)\cdot \mathsf{Im}(\langle v, z\rangle).
\end{aligned}$$

Thus, if $V \in \mathcal{B}'$ and $v \neq iz$ we have

$$\left.\frac{\partial A_j}{\partial V}\right|_* = 0. \tag{4.29}$$

On the other hand, given two vectors $V, W \in \mathcal{B}'$, we have

$$\begin{aligned}
\left.\frac{\partial^2 A_j}{\partial W \partial V}\right|_* &= -2\,\frac{\partial}{\partial W}\mathsf{Re}(\langle v, \xi^j z'\rangle - \langle z, \xi^j \alpha v\rangle)\Big|_* \\
&= 2\,\mathsf{Re}(\langle v, \alpha\xi^j w\rangle + \langle w, \alpha\xi^j v\rangle)\big|_* \\
&= 4\mathsf{Re}(\alpha\xi^j)\mathsf{Re}(\langle v, w\rangle).
\end{aligned}$$

Therefore if $V, W \in \mathcal{B}'$ and $V \neq W$ one has $\left.\frac{\partial^2 A_j}{\partial W\partial V}\right|_* = 0$. Besides, $V \neq W$ implies that either $v \neq iz$ or $w \neq iz$ and by (4.29) either $\left.\frac{\partial A_j}{\partial V}\right|_* = 0$ or $\left.\frac{\partial A_j}{\partial W}\right|_* = 0$, for any j. Hence, by formula (4.28), we have

$$\left.\frac{\partial^2 \widetilde{F}}{\partial W \partial V}\right|_* = 0,$$

for any two distinct vectors V and W of the basis $\mathcal{B}'$. Moreover, if V is such that $v \neq iz$, then

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V \partial V}\right|_* = \sum_{j=0}^{m-1}\left(\frac{\partial^2 A_j}{\partial V \partial V}\cdot \widetilde{F}_j\right)\Bigg|_* = 4\sum_{j=0}^{m-1}(\mathsf{Re}(\alpha\xi^j)\cdot \widetilde{F}_j(*)).$$

For any $0 < j < m-1$ one has

$$\widetilde{F_0}(*) = \widetilde{F}_{m-1}(*) > \widetilde{F}_j(*)$$

and also

$$\mathsf{Re}(\alpha\xi^0) = \mathsf{Re}(\alpha\xi^{m-1}) > 0.$$

Clearly one has $\sum_j \mathsf{Re}(\alpha\xi^j) = 0$ which implies that

$$\sum_{j=1}^{m-2} \mathsf{Re}(\alpha\xi^j) = -2\mathsf{Re}(\alpha\xi^0).$$

Hence,

$$\sum_{j=0}^{m-1} (\mathsf{Re}(\alpha\xi^j) \cdot \widetilde{F}_j(*)) = 2\widetilde{F}_0 \mathsf{Re}(\alpha\xi^0) + \sum_{j=1}^{m-2} (\mathsf{Re}(\alpha\xi^j) \cdot \widetilde{F}_j(*)) > 0.$$

The proof terminates once we prove that

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V \partial V}\right|_* \neq 0,$$

where $V = (v, -\alpha v) \in \mathcal{B}'$ is the basis vector for which $v = iz$. Let

$$I = \{(j,k)|\ 0 \leq j,k \leq m-1,\ k \neq j\}.$$

One has

$$\begin{aligned}
\left.\frac{\partial^2 \widetilde{F}}{\partial V \partial V}\right|_* &= \sum_j \left.\left(\frac{\partial^2 A_j}{\partial V \partial V} + \frac{\partial A_j}{\partial V} \sum_{k \neq j} \frac{\partial A_k}{\partial V} \widetilde{F}_{j,k} \right)\right|_* \\
&= \sum_j \left(4\mathsf{Re}(\alpha\xi^j) + 16\mathsf{Im}(\alpha\xi^j) \cdot \sum_{k \neq j} \left(\mathsf{Im}(\alpha\xi^k) \widetilde{F}_{j,k}(*) \right) \right) \\
&= 16 \cdot \sum_{(j,k) \in I} \mathsf{Im}(\alpha\xi^j) \cdot \mathsf{Im}(\alpha\xi^k) \cdot \widetilde{F}_{j,k}(*) \\
&= 16 \cdot \sum_{(j,k) \in I} \frac{\mathsf{Im}(\alpha\xi^j) \cdot \mathsf{Im}(\alpha\xi^k)}{A_j A_k} \qquad (4.30)
\end{aligned}$$

since $\sum_j \mathsf{Re}(\alpha\xi^j) = 0$. One can easily check that

$$\begin{aligned}
\sum_{(j,k) \in I} \frac{\mathsf{Im}(\alpha\xi^j) \cdot \mathsf{Im}(\alpha\xi^k)}{A_j A_k} &= \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \frac{\mathsf{Im}(\alpha\xi^j) \cdot \mathsf{Im}(\alpha\xi^k)}{A_j A_k} - \sum_{j=0}^{m-1} \left(\frac{\mathsf{Im}(\alpha\xi^j)}{A_j} \right)^2 \\
&= \left(\sum_{j=0}^{m-1} \frac{\mathsf{Im}(\alpha\xi^j)}{A_j} \right)^2 - \sum_{j=0}^{m-1} \left(\frac{\mathsf{Im}(\alpha\xi^j)}{A_j} \right)^2.
\end{aligned}$$

Claim. *For any* $m \geq 2$, *with* $\xi = e^{\frac{2\pi}{m}i}$ *and* $\alpha = e^{\frac{\pi}{m}i}$, *one has*

$$\sum_{j=0}^{m-1} \frac{\mathsf{Im}(\alpha\xi^j)}{A_j} = 0.$$

Proof. Recall that $A_j(z, z') = ||z - \xi^2 z'||^2 = ||1 - \alpha\xi^j||^2$. Let

$$J = \{\alpha\xi^0, \alpha\xi^1, \ldots, \alpha\xi^{m-1}\}$$

and J_+ and J_- be the subsets of J containing the elements with positive and negative imaginary part, respectively. In the picture bellow one has that $J_+ = \{\alpha\xi^0, \alpha\xi^1\}$ and $J_- = \{\alpha\xi^3, \alpha\xi^4\}$. Moreover, there is a reflection $r : J \to J$ on the real line such that $r(J_+) = J_-$.

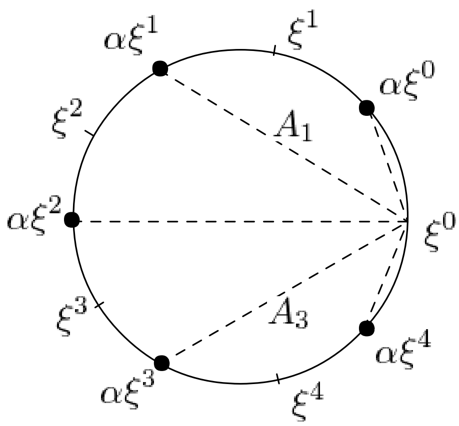

Figure 4.4: The case $m = 5$.

Obviously

$$A_j = ||1 - \alpha\xi^j||^2 = ||1 - r(\alpha\xi^j)||^2 \quad \text{and} \quad \mathsf{Im}(\alpha\xi^j) = -\mathsf{Im}(r(\alpha\xi^j)).$$

Therefore

$$\sum_{j=0}^{m-1} \frac{\mathsf{Im}(\alpha\xi^j)}{A_j} = \sum_{j} \frac{\mathsf{Im}(\alpha\xi^j) + \mathsf{Im}(r(\alpha\xi^j))}{A_j} = 0.$$

$\square$

Resuming from (4.30) one has

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V \partial V}\right|_* = -16 \cdot \sum_{j=0}^{m-1} \left(\frac{\mathsf{Im}(\alpha\xi^j)}{A_j}\right)^2 < 0,$$

since at least one summand in nonzero; *e.g.* $\mathsf{Im}(\alpha)/A_0$. This proves the lemma and also shows that the Bott index at Δ' is 1.

□

The next lemma is the final step to show that $\widetilde{F}$ is a Morse-Bott function.

Lemma 4.2.11. *The Hessian of function $\widetilde{F}$ is nondegenerate on the normal bundle $N\mathcal{V}$.*

Proof. We adopt the notations A_j, $\widetilde{F}_j$ and $\widetilde{F}_{j,k}$ from the proof of Lemma 4.2.9.

Recall that given a point $* = (z, z') \in \mathcal{V}$ in the normal space to the tangent space $T_*\mathcal{V}$ is given by

$$N_*\mathcal{V} = \{\alpha(v_1, v_1') + \beta(v_2, v_2')|\ \alpha, \beta \in \mathbb{R}\},$$

where

$$v_1 = z',\ v_2 = iz',\ v_1' = z \text{ and } v_2' = -iz.$$

As we observed in the proofs of the two previous lemmas,

$$\left.\frac{\partial^2 \widetilde{F}}{\partial W \partial V}\right|_* = \sum_j \left.\left(\frac{\partial^2 A_j}{\partial W \partial V}\widetilde{F}_j + \frac{\partial A_j}{\partial V}\sum_{k\neq j}\frac{\partial A_k}{\partial W}\widetilde{F}_{j,k}\right)\right|_* . \tag{4.31}$$

Set $V_1 = (v_1, v_1')$ and $V_2 = (v_2, v_2')$. Then, for any j, one has

$$\left.\frac{\partial A_j}{\partial V_1}\right|_* = -2\mathsf{Re}\left(\langle v_1, \xi^j z'\rangle + \langle z, \xi^j v_1'\rangle\right) = -2\mathsf{Re}(\xi^{-j} + \xi^{-j}) = -4\mathsf{Re}(\xi^{-j}).$$

and

$$\left.\frac{\partial A_j}{\partial V_2}\right|_* = -2\mathsf{Re}\left(\langle v_2, \xi^j z'\rangle + \langle z, \xi^j v_2'\rangle\right) = -2\mathsf{Re}(i\xi^{-j} + i\xi^{-j}) = 4\mathsf{Im}(\xi^{-j}).$$

On the other hand, for any j and $r, s \in \{1, 2\}$ we have

$$\begin{aligned}\left.\frac{\partial^2 A_j}{\partial V_r \partial V_s}\right|_* &= \left. -2\frac{\partial}{\partial V_r}\mathsf{Re}(\langle v_s, \xi^j z'\rangle + \langle z, \xi^j v_s'\rangle)\right|_* \\ &= -2\mathsf{Re}(\langle v_s, \xi^j v_r'\rangle + \langle v_r, \xi^j v_s'\rangle) \\ &= -2\mathsf{Re}(\xi^{-j}(\langle v_s, v_r'\rangle + \langle v_r, v_s'\rangle)) \\ &= 0,\end{aligned}$$

since $\langle z, z'\rangle = 0$. Hence, by (4.31) we have

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V_r \partial V_s}\right|_* = \left.\sum_{j=0}^{m-1}\left(\frac{\partial A_j}{\partial V_r}\sum_{k\neq j}\frac{\partial A_k}{\partial V_s}\widetilde{F}_{j,k}\right)\right|_*, \tag{4.32}$$

for $r, s = 1, 2$.

Clearly, $\widetilde{F}_{j,k} = 2^{m-2}$ for any j, k. Let

$$I = \{(j,k) |\ 0 \le j, k \le m-1,\ k \neq j\}.$$

Then,

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V_1 \partial V_1}\right|_* = 2^{m-2}\sum_{j=0}^{m-1}\left(4\mathsf{Re}(\xi^j)\sum_{k\neq j}4\mathsf{Re}(\xi^k)\right) = 2^{m+2}\sum_{(j,k)\in I}\mathsf{Re}(\xi^j)\mathsf{Re}(\xi^k),$$

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V_2 \partial V_2}\right|_* = 2^{m-2}\sum_{j=0}^{m-1}\left(4\mathsf{Im}(\xi^j)\sum_{k\neq j}4\mathsf{Im}(\xi^k)\right) = 2^{m+2}\sum_{(j,k)\in I}\mathsf{Im}(\xi^j)\mathsf{Im}(\xi^k)$$

and

$$\left.\frac{\partial^2 \widetilde{F}}{\partial V_1 \partial V_2}\right|_* = 2^{m-2}\sum_{j=0}^{m-1}\left(4\mathsf{Re}(\xi^j)\sum_{k\neq j}4\mathsf{Im}(\xi^k)\right) = 2^{m+2}\sum_{(j,k)\in I}\mathsf{Re}(\xi^j)\mathsf{Im}(\xi^k).$$

Claim. *For any two sequence of numbers $\{r_j\}_{0\le j\le m-1}$ and $\{i_j\}_{0\le j\le m-1}$ such that*

$$\sum_{j=0}^{m-1} r_j = 0 = \sum_{j=0}^{m-1} i_j$$

one has

$$\left(\sum_{(j,k)\in I} r_j r_k\right)\cdot\left(\sum_{(j,k)\in I} i_j i_k\right) - \left(\sum_{(j,k)\in I} r_j i_k\right)^2 \ge 0. \tag{4.33}$$

where the equality holds only if there exists $\lambda \in \mathbb{R}$ such that $r_j = i_j$ for all j.

Proof of the Claim. The proof is based on the Cauchy-Schwarz inequality.

Clearly

$$\begin{aligned}
\sum_{(j,k)\in I} r_j r_k &= \left(\sum_{j=0}^{m-1} r_j\right)^2 - \sum_{j=0}^{m-1} r_j^2 = -\sum_{j=0}^{m-1} r_j^2, \\
\sum_{(j,k)\in I} i_j i_k &= \left(\sum_{j=0}^{m-1} i_j\right)^2 - \sum_{j=0}^{m-1} i_j^2 = -\sum_{j=0}^{m-1} i_j^2, \\
\sum_{(j,k)\in I} r_j i_k &= \left(\sum_{j=0}^{m-1} r_j\right) \cdot \left(\sum_{j=0}^{m-1} i_j\right) - \left(\sum_{j=0}^{m-1} r_j i_j\right) = -\left(\sum_{j=0}^{m-1} r_j i_j\right).
\end{aligned}$$

Hence the statement (4.33) becomes

$$\left(\sum_{j=0}^{m-1} r_j^2\right) \cdot \left(\sum_{j=0}^{m-1} i_j^2\right) - \left(\sum_{j=0}^{m-1} r_j i_j\right) \geq 0.$$

This is precisely the Cauchy-Schwarz inequality in $\mathbb{R}^m$. □

Our intention is to apply this claim with respect to $r_j = \mathsf{Re}(\xi^j)$ and $i_j = \mathsf{Im}(\xi^j)$. By the discussion above one has

$$\begin{aligned}
\det(Hess_* \widetilde{F}) &= \left.\frac{\partial^2 \widetilde{F}}{\partial V_1 \partial V_1}\right|_* \cdot \left.\frac{\partial^2 \widetilde{F}}{\partial V_2 \partial V_2}\right|_* - \left(\left.\frac{\partial^2 \widetilde{F}}{\partial V_1 \partial V_2}\right|_*\right)^2 \\
&= 2^{2m+4} \cdot \left(\left(\sum_{(j,k)\in I} r_j r_k\right) \cdot \left(\sum_{(j,k)\in I} i_j i_k\right) - \left(\sum_{(j,k)\in I} r_j i_k\right)^2\right) \\
&\geq 0.
\end{aligned}$$

Moreover, the vectors $(r_0, \ldots, r_{m-1})$ and $(i_0, \ldots, i_{m-1})$ are clearly independent. Thus we conclude that

$$\det(Hess_* \widetilde{F}) > 0$$

and therefore $Hess_* \widetilde{F}$ in nondegenerate on the normal bundle $N\mathcal{V}$. □

This terminates the proof of Proposition 4.2.8. □

From the proofs of the above lemmas we can conclude the following:

Corollary 4.2.12. *The Bott indices of the navigation function $F : L \times L \to \mathbb{R}$ of Proposition 4.2.1 at the critical submanifolds Δ_L, Δ'_L and V_L are respectively* 0, 1 *and* 2*.*

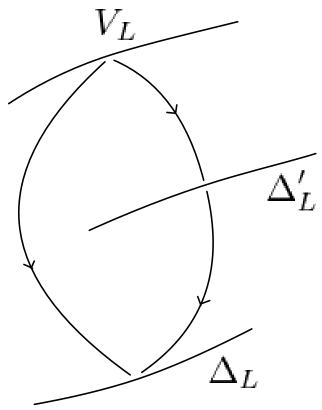

Figure 4.5: Schematic representation of the flow lines.

4.2.3 The case $m = 2$

For any n there is a free $\mathbb{Z}_2$-action on S^n, given by the antipodal map, for which the respective quotient is the real projective space $\mathbb{R}P^n$. Consequently, for $m = 2$ the map $\widetilde{F}$ defined in (4.3) can be extended to even-dimensional spheres. The function $\widetilde{F} : S^n \times S^n \to \mathbb{R}$ given by

$$\widetilde{F}(z, z') = |z - z'|^2 |z + z'|^2 \tag{4.34}$$

and induces a navigation function

$$F : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}.$$

The diagonal $\Delta_P = \{([z], [z']) \in \mathbb{R}P^n \times \mathbb{R}P^n | \, [z] = [z']\}$ is a critical submanifold, corresponding to the level set $F = 0$.

The gradient $\nabla\widetilde{F}$ at a point $(z,z') \in S^n \times S^n - \tilde{\Delta}$ is still defined by (4.8). However we now have

$$\mu(z,z') = \frac{1}{|z-z'|^2} - \frac{1}{|z+z'|^2}.$$

Therefore $\mu(z,z') = 0$ if and only if $|z-z'| = |z+z'|$. This corresponds to a critical submanifold

$$\mathcal{V} = \{([z],[z']) \in \mathbb{R}P^n \times \mathbb{R}P^n |\ \mathsf{Re}(\langle z,z'\rangle) = 0\}$$

where $\langle\cdot,\cdot\rangle$ is the usual hermitian inner product. By Proposition 4.2.3 and 4.2.4, the equation

$$\mu(z,z')z' = \lambda z$$

only has solutions for $\lambda = 0$, *i.e.* when $\langle z,z'\rangle = 0$. Therefore, the submanifold Δ'_L of Proposition 4.2.1 is included in $\mathcal{V}$.

We conclude that the navigation function $F : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}$ induced by (4.34) only has two critical submanifolds.

Proposition 4.2.13. *The navigation function $F : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}$ induced by the map (4.34) has only the two following critical submanifolds:*

1. $\Delta = \{([z],[z']) \in \mathbb{R}P^n \times \mathbb{R}P^n |\ z = z'\}$, *the diagonal;*

2. $\mathcal{V}_P = \{([z],[z']) \in \mathbb{R}P^n \times \mathbb{R}P^n |\ z \perp z'\}$, *the space of pairs of orthogonal lines.*

4.2.4 Comments

Let P denote the projective space $\mathbb{R}P^n$ and L the lens space L_m^{2n-1}. Combining Theorem 4.1.1 with propositions 4.2.1 and 4.2.13 one obtains the following corollary.

Corollary 4.2.14. *Consider the submanifolds* $V_L = \{([z],[z']) \in L \times L : \langle z, z' \rangle = 0\}$ *and* $V_P = \{([z],[z'] \in P \times P|\ \mathsf{Re}(\langle z, z' \rangle) = 0\}$. *One has*

$$\mathsf{TC}(L) \leq 2 + \mathsf{TC}_L(V_L)$$

and

$$\mathsf{TC}(P) \leq 1 + \mathsf{TC}_P(V_P).$$

Proof. By Theorem 4.1.1 the result follows from proving that

$$\mathsf{TC}_L(\Delta_L) = \mathsf{TC}_P(\Delta_P) = 1$$

and

$$\mathsf{TC}_L(\Delta'_L) = 1.$$

For any topological space X, one has $\mathsf{TC}_X(\Delta_X) = 1$ since we can assign the constant path to any point on the diagonal of $X \times X$ thus having a local section of the path-fibration over the diagonal Δ_X. To prove the equality $\mathsf{TC}_L(\Delta'_L) = 1$ we observe that over the 'shifted diagonal' Δ'_L there exists a continuous local section $s : \Delta'_L \to P_{\Delta'_L}L$ of the path fibration $\mathfrak{p} : PL \to L \times L$, where $P_{\Delta'_L}L \subset PL$ is the space of paths in L with endpoints in Δ'_L. In fact, let $s([z],[z'])$ be the path

$$s([z],[z'])(t) = [e^{\frac{\pi}{m}ti} \cdot z], \quad t \in [0,1].$$

Since $[z'] = [e^{\frac{\pi}{m}ti}z]$, we have $s([z],[z'])(0) = [z]$ and $s([z],[z'])(1) = [z']$ and therefore s defines a continuous section over Δ'_L. $\square$

The navigation functions technique discussed in this chapter gives new insight into known results regarding the topological complexity of projective and lens spaces. In [25] Farber, Tabachnikov and Yuzvinsky study the topological complexity of projective spaces and prove the following result.

Theorem 4.2.15. *For any $n \neq 1,3,7$, $\mathsf{TC}(\mathbb{R}P^n) = k+1$ where k is the smallest integer such that the projective space $\mathbb{R}P^n$ admits an immersion into $\mathbb{R}^k$.*

In the same paper (Theorem 7.3) the authors present explicitly $k+1$ motion rules covering $\mathbb{R}P^n \times \mathbb{R}P^n$. In fact, let $U, V \subset \mathbb{R}P^n \times \mathbb{R}P^n$ where U is the set of pairs of lines in $\mathbb{R}^{n+1}$ that form an acute angle and V the complement, *i.e.* the set of pairs of orthogonal lines. The authors point out that U can be covered with a single continuous motion rule $s : U \to P(\mathbb{R}P^n)$ and then show how to use an immersion $\mathbb{R}P^n \to \mathbb{R}^k$ to build k local continuous motion rules covering V. Comparing with Proposition 4.2.13 notice that the gradient flow of the navigation function deformation retracts U onto the diagonal Δ and $V = V_P$. Thus the chosen navigation function, induced from (4.34), is optimal since $\mathsf{TC}(\mathbb{R}P^n) = 1+\mathsf{TC}_{\mathbb{R}P^n}(V_2)$.

It is natural to conjecture if also for lens spaces the inequality of Corollary 4.2.14 is an equality. Recall that

$$\mathsf{TC}_L(V_L) \leq \mathsf{TC}(L) \leq 2 + \mathsf{TC}_L(V_L).$$

The above inequality justifies an interest in estimating $\mathsf{TC}_L(V_L)$. We describe a possible method to determine this value.

The negative gradient flow ϕ_t associated to $F : L \times L \to \mathbb{R}$, given by (4.3), defines a fibration $\eta : \mathcal{U}(V_L) \twoheadrightarrow V_L$ given by

$$\eta([z],[z']) = \lim_{t \to -\infty} \phi_t([z],[z']).$$

With the evolution of the negative gradient flow, as the time parameter t tends to infinity, points in $\mathcal{U}(V_L)$ approach either Δ_L or Δ'_L. Suppose $genus(\eta) = r$. Then V_L admits an open cover $U_1, \ldots, U_r$ such that for every $j \in \{1, \ldots, r\}$ there is a section

$$s_j : U_j \to \mathcal{U}(V_L)$$

of the induced fibration $\iota_j^*\eta : \eta^{-1}(U_j) \to U_j$ with respect to the inclusion map $\iota_j : U_j \to V_L$. Thus V_L can be covered by $2r$ domains $U_1^i, \ldots, U_r^i$, with $i \in \{0,1\}$, defined as follows: $U_j^0 \subset U_j$ is the subset of points in the stable manifold $\mathcal{S}(\Delta_L)$ and $U_j^1 \subset U_j$ the subset of points in the stable manifold of $\mathcal{S}(\Delta_L')$. Clearly each set U_j^i admits a continuous section

$$S_{i,j} : U_j^i \to \eta^{-1}(U_j^i)$$

of the path fibration $\iota_{i,j}^*\mathfrak{p}$, where $\mathfrak{p} : PL \to L \times L$ is the path fibration of L and $\iota_{i,j} : U_j^i \to L \times L$ the inclusion map. Explicitly

$$S_{i,j}([z],[z']) = (r_i \circ s_j)([z],[z']), \quad i \in \{0,1\},\ j \in \{1,\ldots,r\},$$

where $r_0 : \mathcal{S}(\Delta_L) \to L \times L$ and $r_1 : \mathcal{S}(\Delta_L') \to L \times L$ are the natural inclusions.

Let $i : V_L \to L \times L$ denote the inclusion of the critical submanifold V_L. We have just proven that $genus(i^*\mathfrak{p}) \leq 2r = 2genus(\eta)$. By definition of relative topological complexity we have

$$\mathsf{TC}_L(V_L) = genus(i^*\mathfrak{p}) \leq 2genus(\eta).$$

By Corollary 4.2.14

$$\mathsf{TC}(L_m^{2n-1}) \leq 2 + 2genus(\eta). \tag{4.35}$$

Related to $genus(\eta)$ is $genus(\upsilon)$, where υ is the normal bundle to V_L. This is a complex line bundle and studying the respective Chern class may provide the information missing in (4.35).

Gonzalez and Landweber [30] studied the symmetric topological complexity of projective spaces and lens spaces. In that paper the authors successfully related the number $\mathsf{TC}^S(\mathbb{R}P^n)$ with the embedding dimension of $\mathbb{R}P^n$ in a theorem analogous to Theorem 4.2.15 (Theorem 1.3, [25]).

Theorem 4.2.16 ([30]). *For $r > 15$ or $r = 1, 2, 4, 8, 9, 13$, one has*

$$\mathsf{TC}^S(\mathbb{R}P^n) = E(n) + 1,$$

where $E(n)$ stands for the Euclidean embedding dimension of $\mathbb{R}P^n$.

A key element of the proof was the existence of an equivariant deformation retract

$$H : (S^r \times S^r - \widetilde{\Delta}) \times [0,1] \to S^r \times S^r - \widetilde{\Delta},$$

where $\tilde{\Delta} = \{(z, z')|\ z = \pm z'\}$, onto the set of pairs of orthogonal vectors in S^r. The equivariance was with respect to the action generated by coordinate interchange and antipodal mapping. Notice that the gradient flow associated to $\widetilde{F}$, given by (4.34), is equivariant with respect to this action. The authors then claim that a similar deformation retract apparently does not exist in the case of lens spaces (Section 5.1, [30]).

> "Unfortunately, we have not succeeded in obtaining such a connection for larger values of m. The major problem seems to be given by the apparent lack of a suitable equivariant deformation retraction of $L_m^{2n+1} \times L_m^{2n+1} - \Delta L_m^{2n+1}$ that plays the role of $V_{2n+2,2}$[1] (...)"

We observe that Proposition 4.2.1 offers a nice illustration of this phenomenon. The chosen navigation function is the natural generalization of the one chosen for projective spaces. However once $m \geq 3$ a new critical submanifold is formed, which we denoted by Δ'_L. Thus in that case we obtain only an equivariant deformation

[1] $V_{2n+2,2}$ is the set of orthonormal 2-frames in $\mathbb{R}^{2n+2}$. The space V_P mentioned in this section is the $\mathbb{Z}_2 \times \mathbb{Z}_2$-quotient of that space.

retract from $\mathcal{U}(V_L)$ to V_L. Note that from formula (4.8) we can establish that $\mathcal{U}(V_L)$ is the space of pairs $(z, z') \in L_m^{2n-1} \times L_m^{2n-1}$ which are not in the same complex line.

Chapter 5

Topology of Random Right-Angled Artin Groups

It is usual to have a mechanical system for which the parameters are partially unknown. One way to deal with this uncertainty is to treat the parameters of the system as random variables. In spite of the uncertainty associated with the configuration space of a "random" mechanical system, one can often predict some of their topological information; an interesting example is [24], where the authors studied the topology of "random" linkages.

Right-angled Artin groups have connections with some configuration spaces in Robotics. In certain cases the fundamental group of the configuration space $F(\Gamma, n)$ (recall Example 1.1.5) has the structure of a right-angled Artin group; see [28]. On the other hand, right-angled Artin groups can always be viewed as fundamental groups of certain subcomplexes of a n-torus; such complexes have a natural interpretation as configuration spaces of a robot arm with some additional restrictions.

In this chapter we discuss the topology of random right-angled Artin groups with main focus on topological complexity. The contents of this chapter (joint work with M.Farber) were submitted for scientific publication, see [8].

We introduced right-angled Artin groups just before Theorem 2.5.9. In this chapter we are interested in right-angled Artin groups associated to *random graphs* Γ. We adopt one of the basic Erdős - Rényi models of random graphs in which each edge of the complete graph on n vertices is included independently with probability $0 < p < 1$. The probability of obtaining a specific graph Γ by this process is given by

$$\mathbf{P}(\Gamma) \,=\, p^{E_\Gamma}(1-p)^{\binom{n}{2}-E_\Gamma}, \tag{5.1}$$

where E_Γ denotes the number of edges of Γ, see [35].

We will examine statistics of various topological invariants of the group G_Γ associated to a random graph. Each of such invariants is a random function and it is quite natural to ask about its mathematical expectation and distribution function. We will study the asymptotic behaviour of these functions, as n tends to ∞.

Various probabilistic approaches to group theory can be found in [33] and [46].

5.1 Betti numbers of random graph groups

Recall from Definition 2.5.2 that to a finite graph Γ with vertex set V and with the set of edges E is associated the right-angled Artin group (R.A.A.G.)

$$G_\Gamma = |v \in V; vw = wv \quad \text{iff} \quad (v,w) \in E|.$$

There is a well-known construction of an aspherical complex K_Γ with fundamental group G_Γ. Consult [6] and [42] for proofs and more detail.

Let $V = V_\Gamma$ denote the set of vertices of the graph Γ. The torus T^n where $n = |V|$ can be identified with the set of all functions $\phi : V \to S^1$. The support $\mathrm{supp}(\phi) \subset V$ of a function $\phi : V \to S^1$ is defined as the set of vertices $v \in V$ such that $\phi(v) \neq 1$. One defines $K_\Gamma \subset T^n$ to be the set of all functions ϕ such that their support $\mathrm{supp}(\phi)$ generates a complete subgraph of Γ, *i.e.* any two vertices of the support are connected by an edge in Γ. It is well known ([6], [42]) that K_Γ (viewed with the induced topology) is aspherical with fundamental group is G_Γ, *i.e.* K_Γ is the Eilenberg-MacLane complex $K_\Gamma = K(G_\Gamma, 1)$.

We fix the cell decomposition of S^1 consisting of a single 0-cell $1 \in S^1$ and a single 1-cell given by $S^1 - \{1\}$. Clearly T^n inherits a cell decomposition with cells in one-to-one correspondence with subsets of V. In this decomposition $K_\Gamma \subset T^n$ is a cell subcomplex; the cells of K_Γ are in 1-1 correspondence with complete subgraphs of Γ. Namely, given a subset $S \subset V$ one considers the set e_S of all functions $\phi : V \to S^1$ with support S; then e_S is a cell of dimension $|S|$.

The cohomology algebra of K_Γ with integral coefficients is the quotient

$$H^*(K_\Gamma; \mathbf{Z}) \simeq E(v_1, \ldots, v_n)/J_\Gamma \tag{5.2}$$

where $E(v_1, \ldots, v_n)$ is the exterior algebra generated by degree one classes corresponding to the vertices $V = \{v_1, \ldots, v_n\}$ of Γ and the ideal J_Γ is generated by the degree two monomials vw such that the corresponding vertices v, w are not connected by an edge. In particular, any product $v_{i_1} v_{i_2} \ldots v_{i_r}$ vanishes iff the corresponding vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ do not form a complete subgraph of Γ.

One obtains the following well-known facts:

Lemma 5.1.1. *For an integer $r \geq 2$ the r-th Betti number $b_r(G_\Gamma) = b_r(K_\Gamma)$ equals the number of complete subgraphs of size r in Γ. Note that $b_0(G_\Gamma) = 1$ and $b_1(G_\Gamma) =$*

n for any graph Γ.

Lemma 5.1.2. *The expectation of the* r*-th Betti number of the group* G_Γ *of a random graph* Γ*, where* $r \geq 2$*, equals*

$$\mathbf{E}(b_r(G_\Gamma)) = \binom{n}{r} p^{\binom{r}{2}}. \tag{5.3}$$

Proof. We must find the number of complete subgraphs of size r in $\Gamma \in \Omega_n$. For a subset $S \subset \{1, \ldots, n\}$ with $|S| = r$ consider the random variable $I_S : \Omega_n \to \{0, 1\}$ which equals 1 on a graph $\Gamma \in \Omega_n$ iff S forms a complete subgraph in Γ. Then $\mathbf{E}(I_S) = p^{\binom{r}{2}}$ and $\sum_S I_S$ is the number of all complete subgraphs on r vertices. This shows that $\mathbf{E}(\sum_S I_S)$ is as stated. □

Now we assume that r (the dimension) is fixed and p may depend on n. Asymptotically, the expectation of $b_r(G_\Gamma)$ can be written as

$$\mathbf{E}(b_r(G_\Gamma)) \sim \frac{1}{r!} \left[np^{\frac{r-1}{2}} \right]^r.$$

The expectation has a positive limit for $n \to \infty$ if and only if

$$np^{\frac{r-1}{2}} \to c > 0. \tag{5.4}$$

Under this condition the expectation $\mathbf{E}(b_r(G_\Gamma))$ converges to $\dfrac{c^r}{r!}$.

Note that the convergence (5.4) to a positive limit may happen for one dimension r only. Moreover, under the assumption (5.4), the distribution of $b_r : \Omega \to \mathbf{Z}$ converges to the Poisson distribution with expectation

$$\lambda = \frac{c^r}{r!}, \tag{5.5}$$

see below. Theorem 5.1.3 is an interpretation of a theorem of Schürger [44] about complete subgraphs in random graphs.

Theorem 5.1.3. *Fix an integer $r > 1$ and consider the function of r-th Betti number of the associated graph group,*

$$b_r : \Omega_n \to \mathbf{Z}, \quad b_r(\Gamma) = b_r(G_\Gamma),$$

as a random function of a random graph. If the limit (5.4) exists and is positive then for any integer $k = 0, 1, \ldots$ the probability

$$\mathbf{P}(b_r(G_\Gamma) = k)$$

converges (as $n \to \infty$) to

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

where λ is the number defined in (5.5).

In other words, Theorem 5.1.3 claims that the limiting distribution is Poisson with mean λ.

Example 5.1.4. Consider the following examples illustrating the previous result:

(a) Suppose that $r = 2$ and $p = \frac{4}{n^2}$. Then $c = 2$, $\lambda = 2$, and for any integer $k = 0, 1, \ldots$ the probability that $b_2(G_\Gamma) = k$ converges to $\frac{2^k}{e^2 \cdot k!}$ as $n \to \infty$.

(b) As another example, assume that $r = 3$ and $p = \frac{6}{n}$. Then $\lambda = 36$ and the probability that $b_3(G_\Gamma) = k$ converges to $\frac{36^k}{e^{36} \cdot k!}$ as $n \to \infty$.

5.2 Cohomological dimension of G_Γ

It follows from the above previous section that the cohomological dimension of G_Γ equals the size of the maximal clique in Γ; a *clique* in a graph is defined as a maximal complete subgraph. The clique number $\mathrm{cl}(\Gamma)$ of a graph Γ is the maximal order of a clique in Γ.

There are many results in the literature about the clique number of random graphs; we may interpret these results as statements about the cohomological dimension of graph groups build out of random graphs. Matula [40], [41] discovered that for fixed values of p the distribution of the clique number of a random graph is highly concentrated in the sense that almost all random graphs have about the same clique number. These results were developed further by Bollobás and Erdős [4]; consult also the monographs of B. Bollobás [5] and of N. Alon and J. Spencer [2].

Below we restate a result of Matula [41] as a statement about cohomological dimension of random graph groups. Recall that the cohomological dimension of a group G is less than or equal to n, $\mathrm{cd}(G) \leq n$, if for an arbitrary G-module $\mathcal{A}$, the cohomology of G with coefficients in $\mathcal{A}$ vanishes in degrees $k > n$, that is, $H^k(G, \mathcal{A}) = 0$ whenever $k > n$.

Denote

$$z(n,p) = 2\log_q n - 2\log_q \log_q n + 2\log_q(e/2) + 1, \tag{5.6}$$

where $q = p^{-1}$.

Theorem 5.2.1. *Fix an arbitrary $\epsilon > 0$. Then*

$$\lfloor z(n,p) - \epsilon \rfloor \;\leq\; \mathrm{cd}(G_\Gamma) \;\leq\; \lfloor z(n,p) + \epsilon \rfloor, \tag{5.7}$$

asymptotically almost surely (a.a.s). In other words, the probability that a graph $\Gamma \in \Omega_n$ does not satisfy inequality (5.7) tends to zero when n tends to infinity.

Here $\lfloor x \rfloor$ denotes the largest integer not exceeding x. We assume that $\epsilon < 1/2$; then the integers $\lfloor z(n,p) - \epsilon \rfloor$ and $\lfloor z(n,p) + \epsilon \rfloor$ either coincide or differ by 1.

Thus, according Theorem 5.2.1, the cohomological dimension $\mathrm{cd}(G_\Gamma)$ for a random graph Γ takes on one of at most two values depending on n and p, with probability approaching 1 as $n \to \infty$. Moreover, it is known that for most values of n, the clique number is concentrated in a single value ([4]).

The next Lemma is a technical result which will be used later in this chapter.

Lemma 5.2.2. *Fix $\epsilon > 0$ and let $r = \lfloor z(n,p) - \epsilon \rfloor$. Then*

$$r^{-1} \cdot \binom{n}{r} p^{\binom{r}{2}} \to \infty$$

as $n \to \infty$.

Proof. One has $r = \lfloor z(n,p) - \epsilon \rfloor \leq z(n,p) - \epsilon$ and therefore

$$\begin{aligned} p^{\binom{r}{2}} &\geq \left(p^{(z(n,p)-\epsilon-1)/2}\right)^r \\ &= \left(p^{\log_q n - \log_q \log_q n + \log_q(e/2) - \epsilon/2}\right)^r \\ &= \left(\frac{2C \log_q n}{en}\right)^r, \end{aligned}$$

where $C = q^{\epsilon/2} > 1$. On the other hand, using Stirling's formula, we have

$$\binom{n}{r} = c_n \cdot \left(\frac{n}{r}\right)^r \cdot e^r \cdot r^{-1/2}$$

where c_n and c_n^{-1} are bounded. Therefore,

$$r^{-1} \cdot \binom{n}{r} \cdot p^{\binom{r}{2}} \geq r^{-1} c_n \left(\frac{n}{r}\right)^r r^{-1/2} \left(\frac{2C \log_q n}{en}\right)^r = \left(C \cdot \frac{2 \log_q n}{r}\right)^r \cdot r^{-3/2} \cdot c_n \geq C^r \cdot r^{-3/2} \cdot c_n.$$

Clearly, $C^r \cdot r^{-3/2} \cdot c_n$ tends to infinity since $C > 1$. This completes the proof. $\square$

The main result of this chapter states that the inequality (2.4), *i.e.*

$$\mathsf{TC}(X) \leq 2 \dim X + 1,$$

is asymptotically very close to be an equality in the case of Eilenberg - MacLane spaces of random graph groups. However, a specific graph group G can have topological complexity significantly lower than the upper bound

$$\mathsf{TC}(G) \leq 2 \dim K(G, 1) + 1.$$

For example, if Γ is the complete graph on n vertices then the corresponding aspherical space is the n-torus $K_\Gamma = K(G_\Gamma, 1) = S^1 \times \ldots \times S^1$ and $\mathsf{TC}(G_\Gamma) = n + 1$.

5.3 Topological Complexity of random groups

Consider the probability space Ω_n of random graphs on n vertices with probability given by formula (5.1). For any $\Gamma \in \Omega_n$ consider the corresponding Eilenberg-MacLane complex $K_\Gamma = K(G_\Gamma, 1)$ (see section 5.1) and its topological complexity $\mathsf{TC}(K_\Gamma)$.

Theorem 5.3.1. *Fix an arbitrary $0 < \epsilon < 1/2$. Then for any random graph $\Gamma \in \Omega_n$ one has*

$$2 \cdot \lfloor z(n, p) - \epsilon \rfloor + 1 \;\leq\; \mathsf{TC}(K_\Gamma) \;\leq\; 2 \cdot \lfloor z(n, p) + \epsilon \rfloor + 1, \tag{5.8}$$

asymptotically almost surely, where $z(n, p)$ is given by formula (5.6). In other words, probability that a graph $\Gamma \in \Omega_n$ does not satisfy inequality (5.8) tends to zero when n tends to infinity.

It is clear that the integers on the left and on the right of inequality (5.8) differ at most by 2. Hence Theorem 5.3.1 determines the value of the topological complexity $\mathsf{TC}(G_\Gamma)$ for a random graph with ambiguity of at most 2. Comparing with the result of Theorem 5.2.1 we obtain:

Corollary 5.3.2. *For a random graph $\Gamma \in \Omega_n$ one has*

$$2 \cdot \mathrm{cd}(G_\Gamma) - 1 \leq \mathsf{TC}(K_\Gamma) \leq 2 \cdot \mathrm{cd}(G_\Gamma) + 1, \tag{5.9}$$

asymptotically almost surely.

The rest is this section is devoted to the proof of Theorem 5.3.1.

By an (r, r) *bi-clique* in a graph Γ we mean an ordered pair consisting of two disjoint complete subgraphs of Γ on r vertices. To specify an (r, r) *bi-clique* one has to determine an r-element subset S of the set of vertices of Γ and an r-element subset T in the complement $V - S$ such that the induced graphs on S and T are complete.

We have seen in the previous sections that $\mathrm{cd}(G_\Gamma) \geq r$ if and only if Γ contains an r-clique, i.e. a maximal complete subgraph on r vertices. By a theorem of Cohen and Pruidze [9] one has $\mathsf{TC}(K_\Gamma) \geq 2r + 1$ if Γ contains an (r, r) bi-clique.

In the rest of this chapter we set

$$r = \lfloor z(n, p) - \epsilon \rfloor.$$

Theorem 5.3.1 follows once we have shown that a random graph $\Gamma \in \Omega_n$ contains an (r, r) bi-clique a.a.s. The right hand side of the inequality (5.8) follows from the general upper bound $\mathsf{TC}(X) \leq 2 \dim X + 1$ and from the right hand side of (5.7).

Let $r > 0$ be an integer and let $X : \Omega_n \to \mathbf{Z}$ be the random variable that counts the number of (r, r) bi-cliques in random graph. We want to show that $X > 0$ asymptotically almost surely, i.e.

$$\mathbf{P}(X > 0) \to 1, \quad \text{for} \quad n \to \infty. \tag{5.10}$$

The proof of (5.10) will use the second moment method and will be based on the inequality

$$\mathbf{P}(X > 0) \geq \frac{(\mathbf{E}X)^2}{\mathbf{E}(X^2)}, \tag{5.11}$$

see [35], page 54. Thus, our statement follows once we show that

$$\frac{\mathbf{E}(X^2)}{(\mathbf{E}X)^2} \to 1 \quad \text{as} \quad n \to \infty. \tag{5.12}$$

Let S and T be disjoint r-element subsets of the set of vertices of the complete graph K_n and let

$$I_{(S,T)} : \Omega_n \to \{0, 1\}$$

denote the function which equals 1 on a graph $\Gamma \in \Omega_n$ if and only if S and T form a bi-clique in Γ. Then

$$X = \sum_{(S,T)} I_{(S,T)}$$

where the sum is taken over all ordered pairs of disjoint r-element subsets of $\{1, 2, \ldots, n\}$. Note that one obviously has

$$\mathbf{E}(I_{(S,T)}) = p^{2\binom{r}{2}}$$

and thus

$$\mathbf{E}(X) = \binom{n}{r, r} p^{2\binom{r}{2}},$$

where

$$\binom{n}{r, r} = \frac{n!}{r! \cdot r! \cdot (n - 2r)!}$$

denotes the multynomial coefficient. Similarly,

$$X^2 = \sum I_{(S,T)} \cdot I_{(S',T')}. \tag{5.13}$$

Here (S, T) and (S', T') run over all ordered pairs of disjoint r-element subsets of the set of vertices $\{1, \ldots, n\}$.

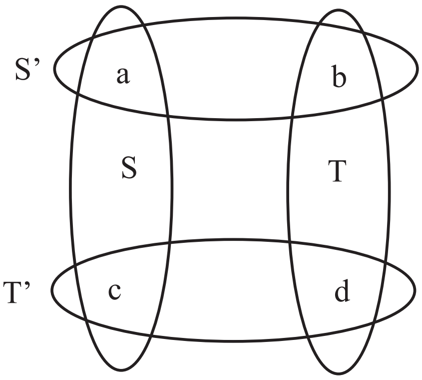

Figure 5.1: A pair of (r, r) bi-cliques.

Denoting

$$a = |S \cap S'|,\ b = |T \cap S'|,\ c = |S \cap T'|,\ d = |T \cap T'|,$$

(see Figure 5.1) we find

$$\mathbf{E}(I_{(S,T)} \cdot I_{(S',T')}) = p^{4\binom{r}{2}-\binom{a}{2}-\binom{b}{2}-\binom{c}{2}-\binom{d}{2}}. \tag{5.14}$$

Therefore taking into account (5.13) one obtains the following expression

$$\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} = \sum_{\alpha \in D} F_\alpha \cdot q^{L(\alpha)} = \sum_{\alpha \in D} T_\alpha. \tag{5.15}$$

Here

$$\alpha = (a, b, c, d) \in \mathbf{Z}^4$$

denotes a vector and D is the set of all vectors $\alpha = (a, b, c, d)$ with nonnegative integer components satisfying the inequalities

$$a + b \le r, \quad a + c \le r, \quad c + d \le r, \quad b + d \le r. \tag{5.16}$$

In formula (5.15) the coefficient F_α is given by

$$F_\alpha = \frac{\binom{r}{a,\, c}\binom{r}{b,\, d}\binom{n-2r}{r-a-b,\, r-c-d}}{\binom{n}{r,r}} \tag{5.17}$$

and

$$L(\alpha) = \binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2}, \qquad q = p^{-1}, \tag{5.18}$$

while

$$T_\alpha = F_\alpha \cdot q^{L(\alpha)}. \tag{5.19}$$

Let $\mathbf{m}(x, y)$ denote $\max\{x, y\}$. Then the inequalities (5.16) can be rewritten in a simple form as

$$\mathbf{m}(a, d) + \mathbf{m}(b, c) \leq r. \tag{5.20}$$

Next we mention the symmetry of the problem. There are two commuting involutions

$$\beta, \gamma : D \to D, \quad \beta^2 = 1 = \gamma^2,$$

where

$$\beta(a) = b, \quad \beta(c) = d, \quad \gamma(a) = c, \quad \gamma(b) = d.$$

These two involutions generate an action of the group $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ on D which preserves both functions T_α and $L(\alpha)$. This action is transitive on the four coordinates.

Recall that our goal is to show that the sum (5.15) tends to 1 as $n \to \infty$. Note that

$$\sum_{\alpha \in D} F_\alpha \;=\; 1 \tag{5.21}$$

for obvious reasons. Observe also that the term F_0 corresponding to $\alpha = (0, 0, 0, 0) \in D$ equals

$$\begin{aligned} F_0 \;&=\; \frac{\binom{n-2r}{r,r}}{\binom{n}{r,r}} = \prod_{k=0}^{2r-1} \left(1 - \frac{2r}{n-k}\right) \\ &\geq\; \left(1 - \frac{2r}{n-2r+1}\right)^{2r} \geq 1 - \frac{4r^2}{n-2r+1}. \end{aligned}$$

Hence we see that $F_0 \to 1$ as $n \to \infty$. Therefore, the sum of all coefficients F_α with $\alpha \neq 0$ tends to zero. However the value of the second factor $q^{L(\alpha)}$ becomes increasingly high when the coordinates of α grow.

As an example, consider the term of (5.15) corresponding to $\alpha = (r, 0, 0, r)$. Then $F_\alpha = \binom{n}{r, r}^{-1}$, $L(\alpha) = 2\binom{r}{2}$ and[1]

$$T_\alpha = F_\alpha \cdot q^{L(\alpha)} = \frac{1}{\binom{n}{r, r}} q^{2\binom{r}{2}} \sim \left[\binom{n}{r} p^{\binom{r}{2}}\right]^{-2}.$$

By Lemma 5.2.2 one obtains

$$r^2 T_{(r,0,0,r)} = o(1). \tag{5.22}$$

As another example consider the term with $\alpha = (r, 0, 0, 0)$. Then

$$T_\alpha = F_\alpha q^{L(\alpha)} = \frac{\binom{n-2r}{r}}{\binom{n}{r, r}} q^{\binom{r}{2}} \sim \left[\binom{n}{r} p^{\binom{r}{2}}\right]^{-1}.$$

In this case we have

$$r T_{(r,0,0,0)} = o(1), \tag{5.23}$$

by Lemma 5.2.2.

The term T_α with $\alpha = (1, 0, 0, 0)$ satisfies

$$T_{(1,0,0,0)} \leq \frac{r^2}{n} \tag{5.24}$$

as one easily checks.

Next we consider T_α with $\alpha = (r-1, 0, 0, 0)$. One has

$$\begin{aligned} T_\alpha &= r \cdot \frac{\binom{n-2r}{1, r}}{\binom{n}{r, r}} q^{\binom{r-1}{2}} \sim r(n-2r) \frac{\binom{n-2r}{r}}{\binom{n}{r, r}} q^{\binom{r-1}{2}} \\ &\sim \frac{r(n-2r)}{\binom{n}{r}} q^{\binom{r-1}{2}} \sim np^{r-1} \cdot \left[\frac{r}{\binom{n}{r} p^{\binom{r}{2}}}\right] \\ &\leq C' np^{r-1} \sim C \frac{\log_q^2 n}{n} \end{aligned}$$

[1]Here the symbol $a_n \sim b_n$ means that the sequences $a_n b_n^{-1}$ and $a_n^{-1} b_n$ are bounded.

for some constants C, C'; here we have used Lemma 5.2.2. Thus, we have the inequality

$$T_{(r-1,0,0,0)} \leq C \cdot \frac{\log_q^2 n}{n}. \tag{5.25}$$

Using similar arguments one obtains

$$T_{(r-1,0,0,r-1)} \leq C'' \cdot \frac{\log_q^4 n}{n^2}, \tag{5.26}$$

where C'' is a constant independent of n.

As a summary of the above discussion of examples we can make the following claim which will be referred to later:

Claim. *If α is either $(1,0,0,0)$, or $(r-1,0,0,0)$, or $(r-1,0,0,r-1)$ then*

$$r^4 T_\alpha = o(1). \tag{5.27}$$

Recall that

$$r = \lfloor 2\log_q n - 2\log_q \log_q n + 2\log_q(e/2) + 1 - \epsilon \rfloor,$$

and in particular $r \leq 2\log_q n$. Fix λ satisfying the inequality

$$0 < \lambda < \frac{1}{1+eq} \tag{5.28}$$

and split the set of all integers in $[0, r]$ into three subsets

$$S_\lambda = \{x \in \mathbf{N}; 0 \leq x \leq (1-\lambda)\log_q n\},$$

$$I_\lambda = \{x \in \mathbf{N}; (1-\lambda)\log_q n < x < (1+\lambda)\log_q n\},$$

$$L_\lambda = \{x \in \mathbf{N}; (1-\lambda)\log_q n \leq x \leq r\}.$$

Integers lying in S_λ, I_λ, and L_λ will be called *"small"*, *"intermediate"* and *"large"*, correspondingly.

Suppose that $\alpha' \in D$ is obtained from $\alpha = (a, b, c, d) \in D$ by increasing of one of the coordinates by 1, say, $\alpha' = (a+1, b, c, d)$. Then the ratio of the corresponding terms of sum (5.15) equals

$$\frac{T_{\alpha'}}{T_\alpha} = \frac{(r-a-b)(r-a-c)}{(a+1)(n-4r+\ell+1)} \cdot q^a,$$

where $\ell = \ell(\alpha) = a+b+c+d$. Clearly, one has

$$n/2 \leq n - 4r + \ell + 1 \leq n,$$

assuming that n is large enough. Hence we obtain

$$A \cdot q^a \leq \frac{T_{\alpha'}}{T_\alpha} \leq 2 \cdot A \cdot q^a \tag{5.29}$$

where

$$A = \frac{(r-a-b)(r-a-c)}{(a+1)n}. \tag{5.30}$$

If $a \in S_\lambda$ is small then $q^a \leq n^{1-\lambda}$, $A \leq \frac{r^2}{n}$ and

$$Aq^a \leq \frac{r^2}{n^\lambda}$$

tends to zero as $n \to \infty$. Hence the ratio which appears in (5.29) is less than 1 for n large enough.

If $a \in L_\lambda$ is large then $q^a \geq n^{1+\lambda}$, $A \geq \frac{1}{2n\log_q n}$ and hence

$$Aq^a \geq \frac{n^\lambda}{2\log_q n}$$

tends to infinity for $n \to \infty$. This gives the following statement:

Lemma 5.3.3. *There exists a constant $N > 0$ such that for all $n \geq N$ the following is true:*

1. *If $\alpha' \in D$ is obtained from $\alpha \in D$ by adding 1 to one of its coordinates which is small (see above) then*

$$T_\alpha > T_{\alpha'}. \tag{5.31}$$

2. *If $\alpha' \in D$ is obtained from $\alpha \in D$ by adding 1 to one of its coordinates which is large then*

$$T_\alpha < T_{\alpha'}. \tag{5.32}$$

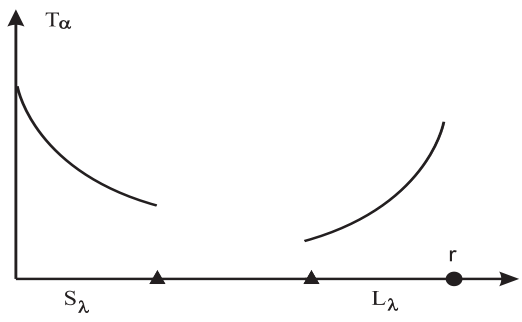

Figure 5.2: Schematic representation of behavior of T_α with respect to small $a \in S_\lambda$ and large $a \in L_\lambda$ coordinates.

Figure 5.2 illustrates Lemma 5.3.3. Next we analyze the case when one increases an intermediate index.

Lemma 5.3.4. *There exists a constant $N > 0$ such that for all $n \geq N$ the following is true: Suppose that $\alpha' = (a+1, b, c, d) \in D$ is obtained from $\alpha = (a, b, c, d) \in D$ by adding 1 to one of its coordinates. If $a \leq r/2$ and $\mathbf{m}(b, c) \notin S_\lambda$, then*

$$T_\alpha > T_{\alpha'}. \tag{5.33}$$

Proof. Without loss of generality we may assume that $a \in I_\lambda$ since the case $a \in S_\lambda$ is covered by Lemma 5.3.3. Then our assumptions imply that $\mathbf{m}(b, c) \in I_\lambda$, and therefore by symmetry we may assume that $b \in I_\lambda$. Our goal is to estimate the value of A given by (5.30). We have

$$a + b > 2(1 - \lambda) \log_q n$$

and since $r < 2 \log_q n$ we obtain

$$r - a - b < 2\lambda \log_q n \tag{5.34}$$

and thus the numerator in (5.30) satisfies

$$(r - a - b)(r - a - c) < 4\lambda \log_q^2 n.$$

To estimate the denominator we observe that $a < r/2$ implies

$$q^a \le \frac{eq}{2} \cdot \frac{n}{\log_q n}.$$

Since $a + 1 \ge (1 - \lambda) \log_q n$ we obtain

$$2Aq^a \le \frac{4\lambda \log_q^2 n}{(1 - \lambda) \log_q n} \cdot \frac{eq}{2} \cdot \frac{n}{\log_q n} \cdot \frac{1}{n} = \frac{4\lambda}{1 - \lambda} \cdot \frac{eq}{2} < 1; \tag{5.35}$$

the last inequality uses our assumption (5.28). This completes the proof of statement (5.33). $\square$

Lemma 5.3.5. *For n sufficiently large and $\alpha = (a, 0, 0, d) \in D$ with $1 \le a \le r - 1$, one has*

$$T_\alpha \le \max\{T_{(1,0,0,d)}, T_{(r-1,0,0,d)}\}. \tag{5.36}$$

Proof. The assertion of the Lemma follows from Lemma 5.3.3 in the case when either $a \in S_\lambda$ or $a \in L_\lambda$. Hence we may assume below that $\alpha = (a, 0, 0, d)$ where $a \in I_\lambda$.

Denote $\alpha' = (a + 1, 0, 0, d)$ and $\alpha'' = (a + 2, 0, 0, d)$. Then

$$\frac{T_{\alpha''} T_\alpha}{T_\alpha^2} = \left(\frac{r - a - 1}{r - a}\right)^2 \cdot \left(\frac{a + 1}{a + 2}\right) \cdot \frac{n - 4r + a + d + 1}{n - 4r + a + d + 2} \cdot q.$$

In the right hand side of this formula the two bracketed factors tend to 1 as $n \to \infty$; besides $q > 1$. Hence for $n > N$ large enough one has

$$\frac{T_\alpha T_{\alpha''}}{T_{\alpha'}^2} > 1. \tag{5.37}$$

This proves that $\log_q(T_\alpha)$ is convex as function of $a \in I_\lambda$. By Lemma 5.3.3 this function increases for $a \in S_\lambda$ and decreases for $a \in L_\lambda$. This implies (5.36). $\square$

Now we can complete the proof of Theorem 5.3.1. Recall that we have to show that the sum $\sum_{\alpha \in D'} T_\alpha$ tends to 0 as $n \to \infty$ where $D' = D - \{(0,0,0,0)\}$. Consider the subset $\tilde{D} \subset D$ consisting of vectors with at least one coordinate equal r. Each $\alpha \in \tilde{D}$ has the form $\alpha = (r, 0, 0, d)$ (up to symmetry) where $d = 0, \dots, r$. Applying Lemma 5.3.5 we obtain that

$$T_\alpha \leq \max\{T_{(r,0,0,0)}, T_{(r,0,0,r)}\}.$$

Since the cardinality of $\tilde{D}$ does not exceed $5r$, we obtain, using (5.22) and (5.23), that

$$\sum_{\alpha \in \tilde{D}} T_\alpha = o(1). \tag{5.38}$$

Each vector $\alpha \in D'$ may have at most two large coordinates. Decompose

$$D' - \tilde{D} = D'_0 \cup D'_1 \cup D'_2,$$

where D'_i denotes the set all vectors in $\tilde{D}$ having exactly i large coordinates, $i = 0, 1, 2$.

Suppose that $\alpha \in D'_2$. Without loss of generality we may assume that a and d are large and b and c are small, i.e. $a, d \in L_\lambda$, $b, c \in S_\lambda$. Applying Lemma 5.3.3 we obtain $T_\alpha \leq T_{(a,0,0,d)}$. Since $a \neq r \neq d$ we may engage Lemma 5.3.5 to obtain

$$T_\alpha \leq \max\{T_{(1,0,0,r-1)}, T_{(r-1,0,0,r-1)}\}. \tag{5.39}$$

Now, taking into account (5.22), (5.23) and (5.27), we obtain

$$\sum_{\alpha\in D_2'} T_\alpha = o(1). \tag{5.40}$$

Consider now the sum $\sum_{\alpha\in D_1'} T_\alpha$. In this case the vector $\alpha = (a, b, c, d)$ contains one large index. Assume that a is large. Then b, c must be small and applying Lemma 5.3.3 and Lemma 5.3.5 we obtain

$$T_\alpha \le T_{(a,0,0,d)} \le T_{(r-1,0,0,d)} \le \max\{T_{(r-1,0,0,0)}, T_{(r-1,0,0,r-1)}\}.$$

Now (5.27) implies that

$$\sum_{\alpha\in D_1'} T_\alpha = o(1). \tag{5.41}$$

Next we show that for any $\alpha \in D_0'$ one has

$$T_\alpha \le \max\{T_{(1,0,0,0)}, T_{(r-1,0,0,0)}, T_{(r-1,0,0,r-1)}\} \tag{5.42}$$

which in view of (5.27) would imply that

$$\sum_{\alpha\in D_0'} T_\alpha = o(1). \tag{5.43}$$

The combination of (5.38), (5.40), (5.41) and (5.43) gives Theorem 5.3.1.

To prove (5.42) consider $\alpha = (a, b, c, d) \in D_0'$. Note that coordinates a, b, c, d can be either small or intermediate. Assume first that all coordinates a, b, c, d are small. Then $T_\alpha \le T_{(1,0,0,0)}$ (by Lemma 5.3.3) implying (5.42).

Suppose now that exactly one of the coordinates of α is intermediate. If a is intermediate and b, c, d are small then

$$T_\alpha \le T_{(a,0,0,0)} \le \max\{T_{(1,0,0,0)}, T_{(r-1,0,0,0)}\}$$

(by Lemma 5.3.3 and Lemma 5.3.5) proving (5.42).

Consider the case where two coordinates of α are intermediate. Taking into account symmetry (the action of G on D, see above), this case can be subdivided into two subcases: (i) a and b are intermediate and (ii) a and d are intermediate. In the subcase (i), since $a + b \le r$, either $a \le r/2$, or $b \le r/2$ and we may apply Lemma 5.3.4. Assuming that $a \le r/2$ we obtain

$$T_\alpha \le T_{(0,b,0,0)} \le \max\{T_{(1,0,0,0)}, T_{(r-1,0,0,0)}\},$$

implying (5.42). In the subcase (ii), we know that b, c are small hence $T_\alpha \le T_{(a,0,0,d)}$ and application of Lemma 5.3.5 gives (5.42).

In the remaining case when $\alpha \in D_0'$ has three or four intermediate indices we know that at least two of these indices are $\le r/2$ and by Lemma 5.3.4 one has

$$T_\alpha \le T_{\alpha'}$$

where α' is obtained from α by replacing by zeros two coordinates which were $\le r/2$. To estimate $T_{\alpha'}$ one applies Lemma 5.3.5 leading again to (5.42). This completes the proof of Theorem 5.3.1. $\square$

Chapter 6

Conclusions

In this thesis we discussed the homotopy invariant $\mathsf{TC}(X)$; the topological complexity of a space X. The original results of this thesis were presented in Chapters 3, 4 and 5.

In Chapter 3 we presented new upper bounds for spaces with 'small' fundamental groups. From Theorem 3.1.1 we have the upper bound $\mathsf{TC}(G_k(\mathbb{R}^{n+k})) \leq 2kn$, where $G_k(\mathbb{R}^{n+k})$ denotes the real Grassmannian, the manifold of n-dimensional real vector subspaces of $\mathbb{R}^{n+k}$. Moreover, it follows from theorems 3.1.1 and 3.1.2 that when $\mathsf{cat}(G_k(\mathbb{R}^{n+k}))$ is not maximal then $\mathsf{TC}(G_k(\mathbb{R}^{n+k})) \leq 2kn - 1$. In [3], the author shows that in some cases $\mathsf{cat}(G_k(\mathbb{R}^{n+k})) = \dim(G_k(\mathbb{R}^{n+k})) + 1 = nk + 1$. Notice that by applying the upper bound given by Proposition 2.2.2 only allows to establish the general dimensional upper bound $\mathsf{TC}(G_k(\mathbb{R}^n)) \leq 2kn + 1$. We warn the reader of a mistake in [43]. There it was wrongly assumed that $\mathsf{TC}(X) = \mathsf{cat}(X \times X)$ (Theorem 1.8, [43]). For example $\mathsf{TC}(S^1) = 2 \neq 3 = \mathsf{cat}(S^1 \times S^1)$. This compromises the results stated in [43]. It would be interesting to obtain more results relating the algebraic properties of the fundamental group of a space X with the number $\mathsf{TC}(X)$.

In Chapter 4 we introduced and studied a class of navigation functions on projective and lens spaces. This study is incomplete and should be addressed in future work. The goal is to obtain new upper bounds for the (symmetric and nonsymmetric) topological complexity of lens spaces. The function $\widetilde{F}$ defined in (4.2) can be written as

$$\prod_{g\in\mathbb{Z}_m} A(z, g\cdot z'),$$

where $A(z,z') = ||z - z'||^2$ and $g\cdot z'$ is given by the product $\xi^g z'$. This method for creating a navigation function can be generalized for other spaces, such as the Klein bottle. Denote by K and $T = S^1 \times S^1$ the Klein bottle and the two dimensional torus. Let $\phi : T \to \mathbb{R}$ be the involution on T given by $\phi(z_1, z_2) = (-z_1, \bar{z_2})$. The Klein bottle can be obtained by the quotient $K = T/\phi$. One can then define a ϕ-invariant map $\widetilde{G} : T \times T \to \mathbb{R}$ given by

$$\widetilde{G}(z, z') = A(z, z') \cdot A(z, \phi(z')),$$

where $A(z,z') = ||z - z'||^2$. This can help to solve the open question regarding the precise value of $\mathsf{TC}(K)$.

In Chapter 5 we estimated the topological complexity of random right-angled Artin groups (also called graph groups). These are groups induced from random graphs. We showed that with probability tending to one, the topological complexity of a random graph group is concentrated in at most three values.

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